Physical Principles in Biology
Biology 3550
Fall 2018

Lecture 10:

Random Walks in One Dimension

Wednesday, 12 September 2018

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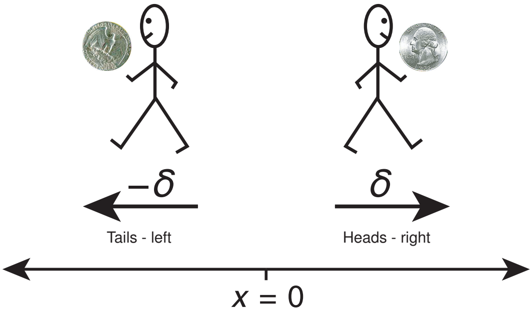
Announcements

- Quiz 2: Wednesday, 19 Sept.
- Problem Set 2 is now posted on course web page.
  Due Monday, 24 Sept.
  Start early!

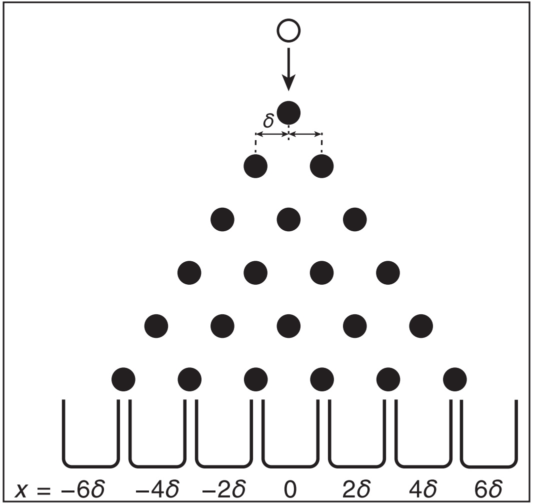
A Random Walk in One Dimension



1. Start at position $x = 0$.
2. Flip coin.
   - Heads, take step of length δ to the right.
   - Tails, take step of length δ to the left.
3. Repeat 2 another $(n - 1)$ times.

- Final position after n steps is x_n.
- Generally expect a distribution of x_n if the random walk is repeated a large number, N, of times.

Like a Plinko, with variable x



- x represents the position of the bucket, relative to the central bucket.

What Do We Know About x_n, the End Point?

- The maximum value of x_n is δn.
- The minimum value of x_n is $-\delta n$.
- If we repeat the random walk many times, the distribution of x_n will be binomial.
- But, if n is very large, calculating the binomial distribution will be difficult!

Clicker Question #1

What is the largest value of n for which your calculator can calculate $n!$?

A) 1–25

B) 26–50

C) 51–75

D) 76–100

E) > 100

Calculate The Average Final Position (The Expected Value of x_n)

- For a single random walk, the final position will be:

$$x_n = \sum_{i=1}^{n} \delta_i$$

where i is the step number, and δ_i is either $+\delta$ or $-\delta$, with probabilities $p_{+\delta}$ and $p_{-\delta}$.

- For each step, δ_i is a random variable, with an expected value, $E(\delta_i)$:

$$\begin{aligned} E(\delta_i) &= \delta p_{+\delta} - \delta p_{-\delta} \\ &= \delta p_{+\delta} - \delta(1 - p_{+\delta}) \\ &= \delta p_{+\delta} - \delta + \delta p_{+\delta} \\ &= 2\delta p_{+\delta} - \delta = \delta\left(2p_{+\delta} - 1\right) \end{aligned}$$

Clicker Question #2

If the random-walk step size is 0.5 m, and the probability of a forward step, $p_{+\delta}$, is 0.3, what is the expected value for the displacement in a single step, $E(\delta_i)$?

A) -0.5 m

B) -0.2 m

C) 0 m

D) 0.2 m

E) 0.5 m

$$
\begin{aligned}
E(\delta_i) &= \delta p_{+\delta} - \delta p_{-\delta} \\
&= 0.5\,\mathrm{m} \cdot 0.3 - 0.5\,\mathrm{m} \cdot 0.7 \\
&= 0.5\,\mathrm{m}(0.3 - 0.7) = -0.2\,\mathrm{m}
\end{aligned}
$$

Calculating The Expected Value of x_n

- An important theorem: If x and y are two independent random variables, then the expected value of the sum is calculated as:

$$E(x + y) = E(x) + E(y)$$

- Since:

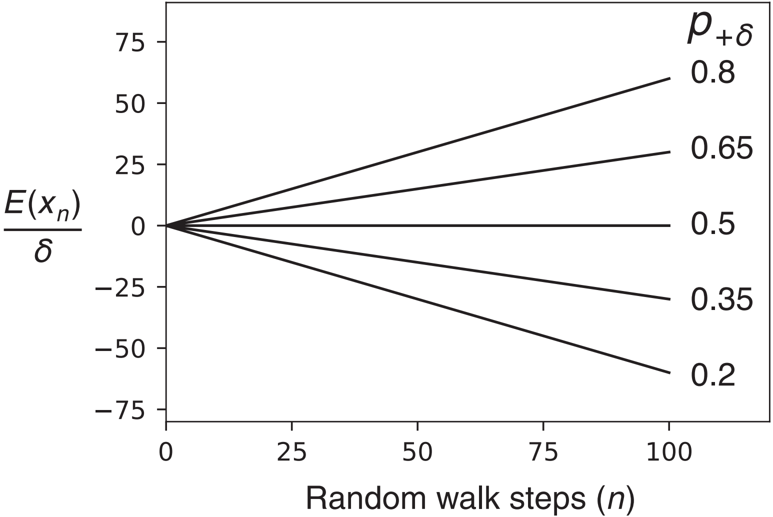
$$x_n = \sum_{i=1}^{n} \delta_i$$

The expected value of x_n is calculated as:

$$E\left(x_n\right) = \sum_{i=1}^{n} E(\delta_i) = nE(\delta_i)$$

$$= n\delta\left(2p_{+\delta} - 1\right)$$

Expected Value of x_n for a One-dimensional Random Walk



Some Different Kinds of Average

For N random walks of n steps each:

- The mean:

$$\langle x_n \rangle = \frac{1}{N} \sum_{j=1}^{N} x_{n,j}, \text{ for large } N$$

  Angle brackets, $\langle \ \rangle$, indicate average over a large sample.

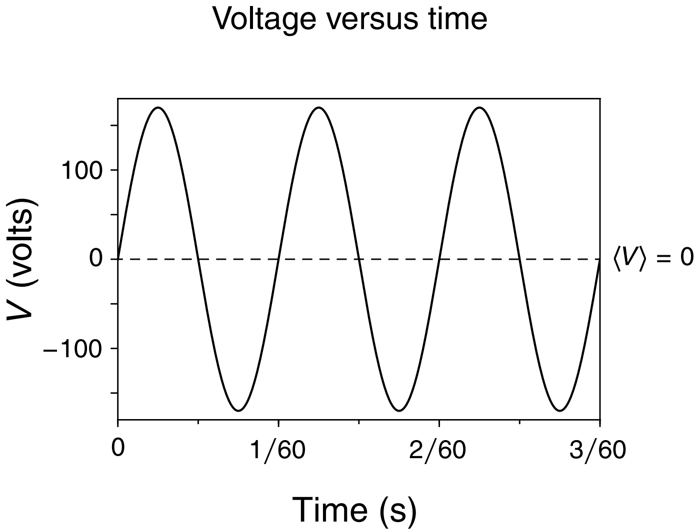
- The mean-square:

$$\langle x_n^2 \rangle = \frac{1}{N} \sum_{j=1}^{N} x_{n,j}^2$$

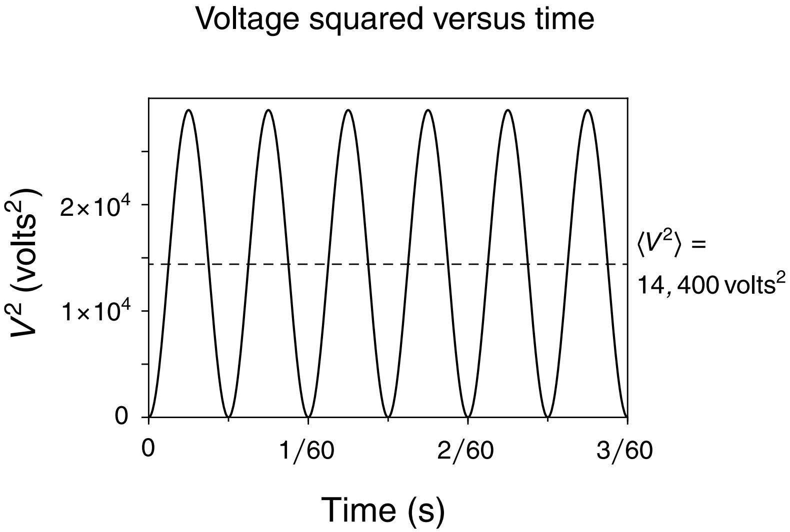
- The root-mean-square (RMS):

$$\mathrm{RMS}(x_n) = \sqrt{\langle x_n^2 \rangle} = \sqrt{\frac{1}{N} \sum_{j=1}^{N} x_{n,j}^2}$$

An Application of Mean-square and Root-mean-square Averages: Household Power (US)

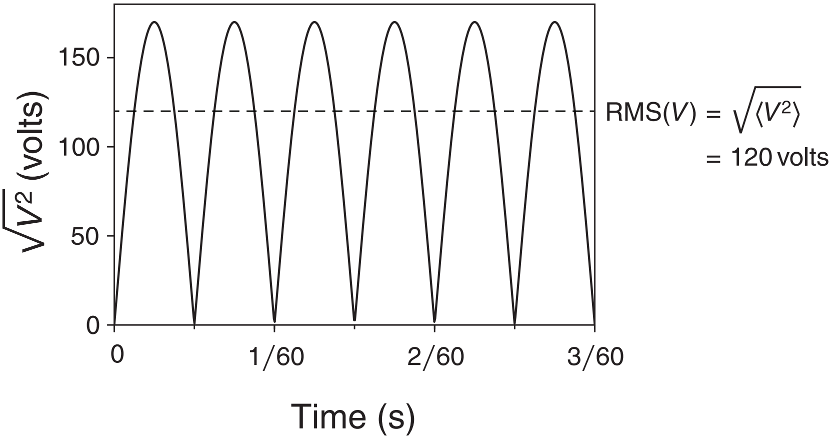


An Application of Mean-square and Root-mean-square Averages: Household Power (US)



An Application of Mean-square and Root-mean-square Averages: Household Power (US)

$\sqrt{V^2}$ versus time



Clicker Question #3

For the numbers: $-4, 2, -3, 1, 5,$
Calculate the root-mean-square average

A) ~ 0.2

B) ~ 1.5

C) ~ 2.9

D) ~ 3.3

E) ~ 4.8

$$\text{RMS} = \sqrt{\frac{-4^2 + 2^2 + -3^2 + 1^2 + 5^2}{5}} = \sqrt{\frac{16 + 4 + 9 + 1 + 25}{5}} = \sqrt{\frac{55}{5}} = \sqrt{11}$$

Calculating the Mean-Square Displacement for a 1-d Random Walk

- For a single random walk, the final position will be:

$$x_n = \sum_{i=1}^{n} \delta_i$$

  where i is the step number, and δ_i is either $-\delta$ or $+\delta$, with equal probability (if the coin is fair), for each step.

- We can also express x_n in terms of the position after the next-to-last step, x_{n-1}:

$$x_n = x_{n-1} + \delta_n$$

  Something tricky is coming up!

Calculating The Mean-Square Displacement

- If we do a large number, N, of random walks, the mean-square displacement, $\langle x \rangle$, will be:

$$\langle x_n^2 \rangle = \frac{1}{N} \sum_{j=1}^{N} x_{n,j}^2 = \frac{1}{N} \sum_{j=1}^{N} \left(\sum_{i=1}^{n} \delta_{j,i} \right)^2$$

  where j is the random walk number, and $\delta_{j,i}$ is the displacement for step i of walk j.

- We can also write the mean-square average as:

$$\langle x_n^2 \rangle = \frac{1}{N} \sum_{j=1}^{N} \left(x_{(n-1),j} + \delta_{j,n} \right)^2$$

$$= \frac{1}{N} \sum_{j=1}^{N} \left(x_{(n-1),j}^2 + 2x_{(n-1),j}\delta_{j,n} + \delta_{j,n}^2 \right)$$

  $\delta_{n,j}$ is the change in position for the last step in random walk j.

Calculating The Mean-Square Displacement

- Following from the previous slide:

$$\langle x_n^2 \rangle = \frac{1}{N}\sum_{j=1}^{N}\left(x_{(n-1),j}^2 + 2x_{(n-1),j}\delta_{j,n} + \delta_{j,n}^2\right)$$

$$= \frac{1}{N}\sum_{j=1}^{N} x_{(n-1),j}^2 + \frac{1}{N}\sum_{j=1}^{N}(2x_{(n-1),j}\delta_{j,n} + \frac{1}{N}\sum_{j=1}^{N}\delta_{j,n}^2$$

$$= \langle x_{n-1}^2 \rangle + \langle 2x_{n-1}\delta_n \rangle + \langle \delta_n^2 \rangle$$

- For each walk, δ_n is either $+\delta$ or $-\delta$, with equal probability (for an unbiased walk), and is independent of the position before the last step, x_{n-1}.
- The average of $2x_{n-1}\delta_n$ over N walks, $\langle 2x_{n-1}\delta_n \rangle$, is expected to be 0.

Calculating The Mean-Square Displacement

- From the previous slide:

$$\langle x_n^2 \rangle = \langle x_{n-1}^2 \rangle + \langle \delta_n^2 \rangle$$

- Following the same logic:

$$\langle x_{n-1}^2 \rangle = \langle x_{n-2}^2 \rangle + \langle \delta_{n-1}^2 \rangle$$

- and

$$\begin{aligned} \langle x_n^2 \rangle &= \langle x_{n-2}^2 \rangle + \langle \delta_{n-1}^2 \rangle + \langle \delta_n^2 \rangle \\ &= \langle x_{n-2}^2 \rangle + 2\langle \delta^2 \rangle \end{aligned}$$

Calculating The Mean-Square Displacement

- Continuing in the same way:

$$\begin{aligned}\langle x_n^2 \rangle &= \langle x_{n-2}^2 \rangle + 2\langle \delta^2 \rangle \\ &= \langle x_{n-3}^2 \rangle + \langle \delta_{n-2}^2 \rangle + 2\langle \delta^2 \rangle \\ &= \langle x_{n-3}^2 \rangle + 3\langle \delta^2 \rangle\end{aligned}$$

$$\langle x_n^2 \rangle = \langle x_{n-4}^2 \rangle + 4\langle \delta^2 \rangle$$

- and so on, until we have:

$$\begin{aligned}\langle x_n^2 \rangle &= \langle x_1^2 \rangle + (n-1)\langle \delta^2 \rangle \\ &= \langle x_0^2 \rangle + n\langle \delta^2 \rangle \\ &= n\langle \delta^2 \rangle\end{aligned}$$

- A derivation based on recursion!