Physical Principles in Biology
Biology 3550
Fall 2018

Lecture 12:

From One-dimensional to Two-dimensional Random Walks

Monday, 17 September 2018

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Important Results for Random Walks So Far

- The average displacement in a single step:

$$\langle \delta \rangle = \delta(2p_{+\delta} - 1)$$

- The average value of x_n over a large number of random walks:

$$\langle x_n \rangle = n\delta(2p_{+\delta} - 1)$$

- The average value of x_n^2 over a large number of random walks:

$$\langle x_n^2 \rangle = n\langle \delta^2 \rangle$$

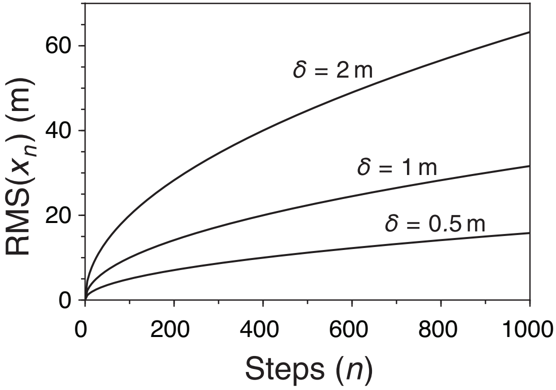
if $\langle \delta \rangle = 0$

- If there are only two possible values for δ_i

$$\langle x_n^2 \rangle = n\delta^2 \text{ and } \mathrm{RMS}(x_n) = \sqrt{n} \times \sigma$$

The Root-mean-square Displacement for a One-dimensional Random Walk

$$\mathrm{RMS}(x_n) = \sqrt{\langle x_n^2 \rangle} = \sqrt{n} \times \delta$$



- This is THE most important thing to remember about random walks!

Clicker Question #1

For a one-dimensional random walk of 100 steps, of length 0.25 m, what is the total distance traveled, as measured by a pedometer (or a FitBit)?

A) 2.5 m

B) 5 m

C) 7.9 m

D) 25 m

E) 50 m

F) 79 m

$$\text{Total distance} = n\delta$$
$$= 100 \times 0.25\,\text{m} = 25\,\text{m}$$

Clicker Question #2

For a one-dimensional random walk of 100 steps, of length 0.25 m, what is the expected RMS distance between the starting and ending points?

A) 2.5 m
B) 5 m
C) 7.9 m
D) 25 m
E) 50 m
F) 79 m

$$\begin{aligned}\mathrm{RMS}(x_n) &= \sqrt{n}\delta \\ &= \sqrt{100} \times 0.25\,\mathrm{m} = 10 \times 0.25\,\mathrm{m} = 2.5\,\mathrm{m}\end{aligned}$$

Clicker Question #3

For a one-dimensional random walk of 10 steps, of length 2.5 m, what is the total distance traveled?

A) 2.5 m

B) 5 m

C) 7.9 m

D) 25 m

E) 50 m

F) 79 m

$$\text{Total distance} = n\delta$$
$$= 10 \times 2.5\,\text{m} = 25\,\text{m}$$

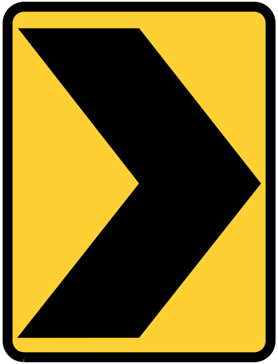
Clicker Question #4

For a one-dimensional random walk of 10 steps, of length 2.5 m, what is the expected RMS distance between the starting and ending points?

A) 2.5 m
B) 5 m
C) 7.9 m
D) 25 m
E) 50 m
F) 79 m

$$\begin{aligned} \mathrm{RMS}(x_n) &= \sqrt{n}\delta \\ &= \sqrt{10} \times 2.5\,\mathrm{m} \approx 3.162 \times 2.5\,\mathrm{m} \approx 7.9\,\mathrm{m} \end{aligned}$$

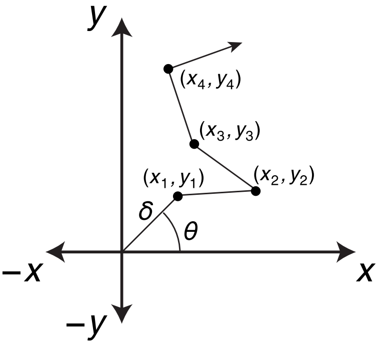
Warning!



Direction Change

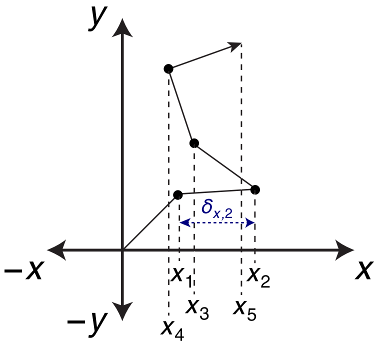
Two-dimensional Random Walks

A Random Walk in Two Dimensions



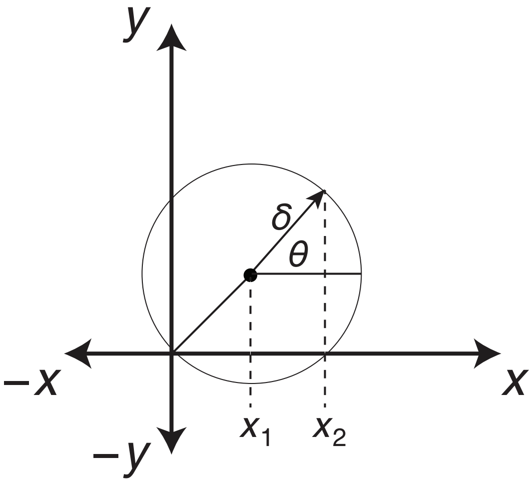
1. Start at (x, y) coordinates (0,0).
2. Choose a random direction, defined by the angle θ from the x-axis.
3. Move distance δ in the chosen direction.
4. Repeat 2 and 3 another $(n - 1)$ times.

A Random Walk in Two Dimensions



- x-coordinates represent a random walk along the x-axis.
- Can also describe a random walk along the y-axis (or any other direction).
- What are $\langle x_n \rangle$, $\langle x_n^2 \rangle$ and $\mathrm{RMS}(x_n)$?
- The change in x with each step, $\delta_{x,i}$, is not discrete!
- $\langle \delta_x^2 \rangle \neq \delta^2$

Why $\langle \delta_x^2 \rangle$ is not Equal to δ^2 for a Two-dimensional Random Walk



- For each random walk step:

  $\delta_{x,i} = x_i - x_{i-1}$

- If $\theta = 0$, then:

  $\delta_{x,i} = \delta$

  $\delta_{x,i}^2 = \delta^2$

- If $\theta = \pi\,\text{rad}$, then:

  $\delta_{x,i} = -\delta$

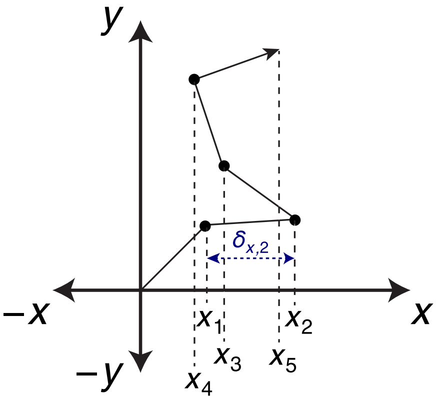
  $\delta_{x,i}^2 = \delta^2$

- For most values of θ:

  $\left|\delta_{x,i}\right| < \delta$

  $\delta_{x,i}^2 < \delta^2$

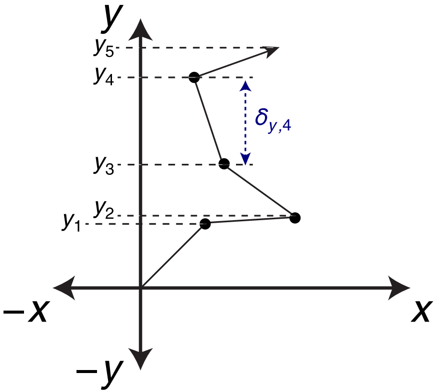
The Random Walk Along the x-axis



- $\langle x_n \rangle = 0$
- $\langle x_n^2 \rangle = n \langle \delta_x^2 \rangle$
- $\langle \delta_x^2 \rangle < \delta^2$
- $\langle x_n^2 \rangle < n\delta^2$
- Mean-square displacement along the x-axis is less than for a random walk constrained to the x-axis, with the same step size, δ.

  By how much?

The Random Walk Along the y-axis



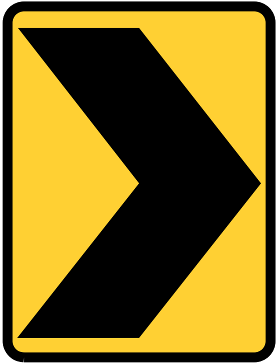
- $\langle y_n \rangle = 0$
- $\langle y_n^2 \rangle = n \langle \delta_y^2 \rangle$
- If all values of θ are equally probable:

  $\langle \delta_x^2 \rangle = \langle \delta_y^2 \rangle = \langle \delta_{x,y}^2 \rangle$
- The average x- and y-projections should be the same:

  $\langle x_n \rangle = \langle y_n \rangle = 0$

  $\langle x_n^2 \rangle = \langle y_n^2 \rangle = n \langle \delta_{x,y}^2 \rangle$
- BUT: $\langle \delta_{x,y}^2 \rangle < \delta^2$

Warning!

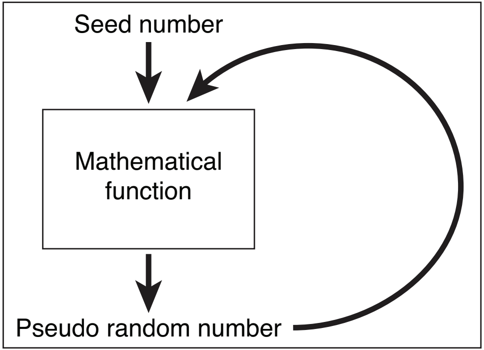


Direction Change

Simulating Random Walks

Simulating Random Processes with a Computer

- Computers aren't supposed to do things at random!
- But, we often ask them to!
- Pseudo-random number generators

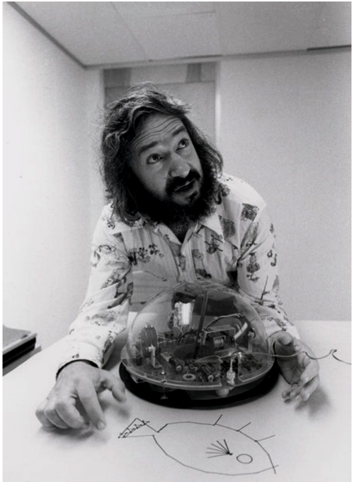


- Function has to be carefully designed so that numbers "look random".

Where Does the Seed Number Come From?

- A user-specified number, to generate a predictable set of pseudo-random numbers. Useful for simulations.
- The computer's clock. Very common method.
- A truly random physical process:
  - Radioactive decay.
    `https://www.fourmilab.ch/hotbits/`
  - A lava lamp!
    `https://en.wikipedia.org/wiki/Lavarand`
  - Electronic noise.
    `https://en.wikipedia.org/wiki/Hardware_random_number_generator`
- Good random numbers are are becoming more important every day!

Seymour Papert 1928–2016



- Education researcher
- Computer scientist
- Co-developer of the Logo computer language
- Turtle graphics

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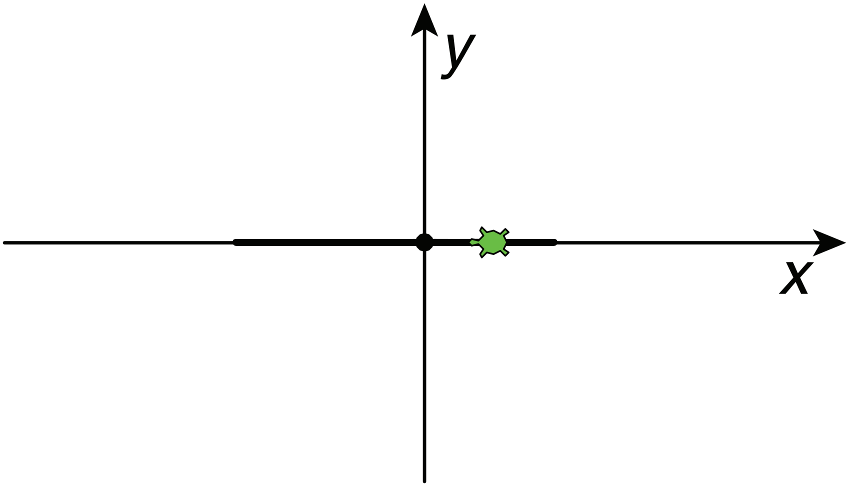
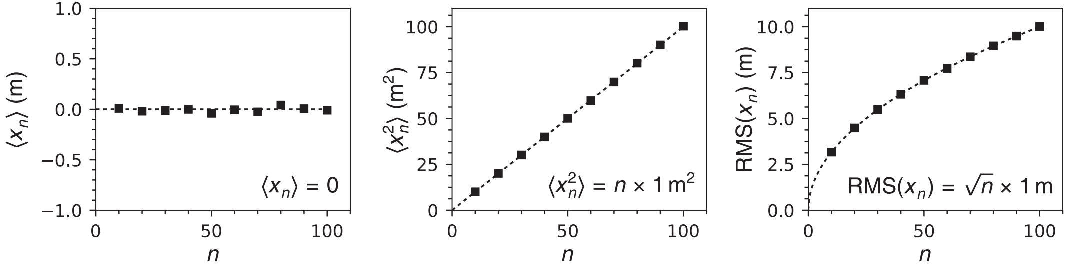
http://www.nytimes.com/2016/08/02/technology/seymour-papert-88-dies-saw-educations-future-in-computers.html?_r=0
http://el.media.mit.edu/logo-foundation/

A One-dimensional Random Walk by a Turtle



Results from Simulated One-dimensional Random Walks



- 100,000 simulated random walks for each value of n, from 10 to 100.
- Unbiased random walks: $p_{+\delta} = p_{-\delta} = 0$
- Step length: $\delta = 1\,\text{m}$
- Excellent agreement between theory and computational experiment!