Physical Principles in Biology
Biology 3550
Fall 2018

Lecture 13:
Two-dimensional Random Walks

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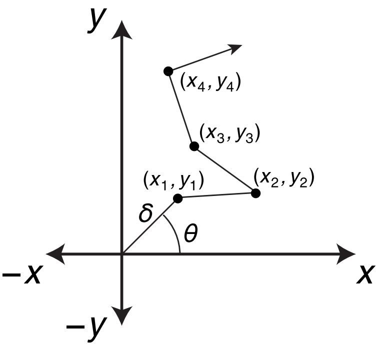
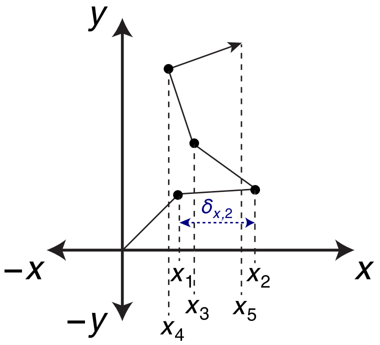
A Random Walk in Two Dimensions



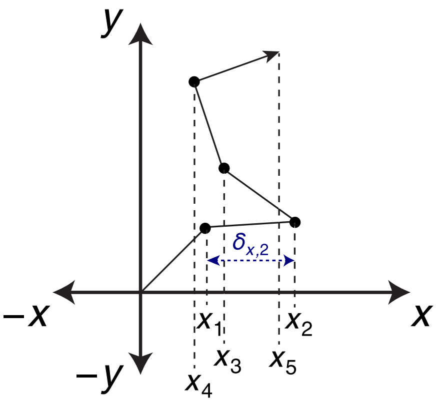
1. Start at (x, y) coordinates (0,0).
2. Choose a random direction, defined by the angle θ from the x-axis.
3. Move distance δ in the chosen direction.
4. Repeat 2 and 3 another $(n - 1)$ times.

A Random Walk in Two Dimensions



- x-coordinates represent a random walk along the x-axis.
- Can also describe a random walk along the y-axis (or any other direction).
- What are $\langle x_n \rangle$, $\langle x_n^2 \rangle$ and $\mathrm{RMS}(x_n)$?
- The change in x with each step, $\delta_{x,i}$, is not discrete!
- $\langle \delta_x^2 \rangle \neq \delta^2$

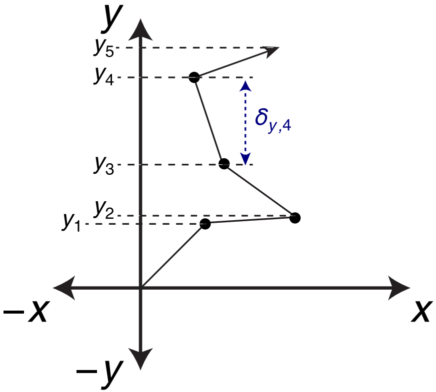
The Random Walk Along the x-axis



- $\langle x_n \rangle = 0$
- $\langle x_n^2 \rangle = n \langle \delta_x^2 \rangle$
- $\langle \delta_x^2 \rangle < \delta^2$
- $\langle x_n^2 \rangle < n\delta^2$
- Mean-square displacement along the x-axis is less than for a random walk constrained to the x-axis, with the same step size, δ.

  By how much?

The Random Walk Along the y-axis



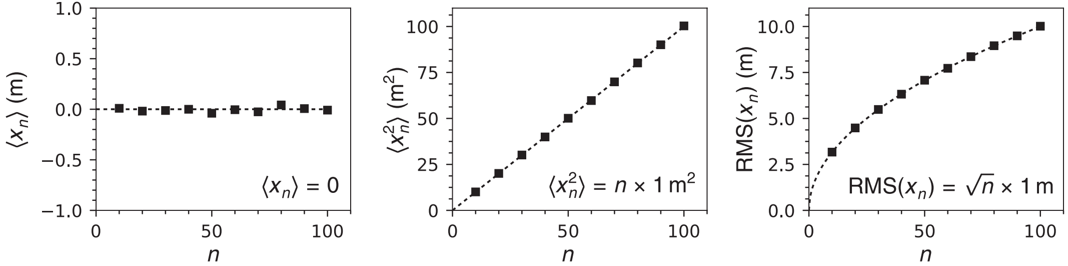
- $\langle y_n \rangle = 0$
- $\langle y_n^2 \rangle = n \langle \delta_y^2 \rangle$
- If all values of θ are equally probable:

  $\langle \delta_x^2 \rangle = \langle \delta_y^2 \rangle = \langle \delta_{x,y}^2 \rangle$
- The average x- and y-projections should be the same:

  $\langle x_n \rangle = \langle y_n \rangle = 0$

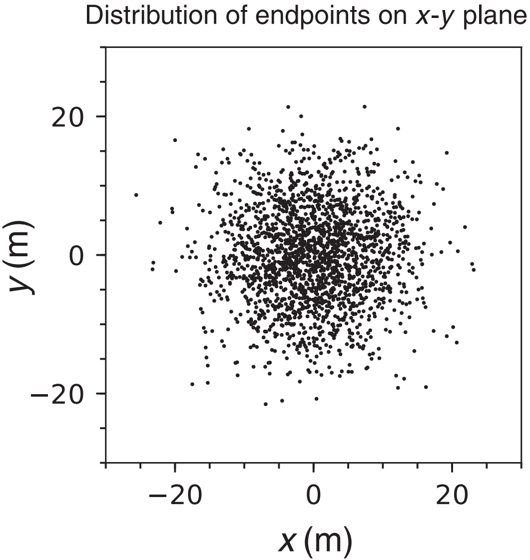
  $\langle x_n^2 \rangle = \langle y_n^2 \rangle = n \langle \delta_{x,y}^2 \rangle$
- BUT: $\langle \delta_{x,y}^2 \rangle < \delta^2$

Results from Simulated One-dimensional Random Walks



- 100,000 simulated random walks for each value of n, from 10 to 100.
- Unbiased random walks: $p_{+\delta} = p_{-\delta} = 0$
- Step length: $\delta = 1\,\mathrm{m}$
- Excellent agreement between theory and computational experiment!

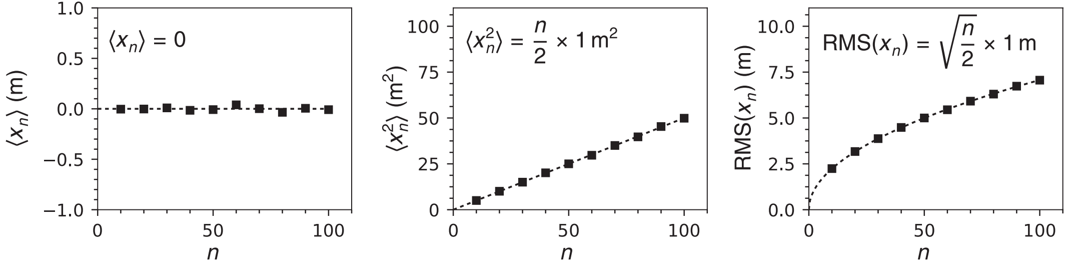
Results from Simulated Two-dimensional Random Walks



- 2,000 simulated random walks
- 100 steps
- Unbiased random walks: Turns in any direction are equally likely.
- Step length: $\delta = 1$ m
- Randomness is lumpy! (Unless N is very large)

Results from Simulated Two-dimensional Random Walks

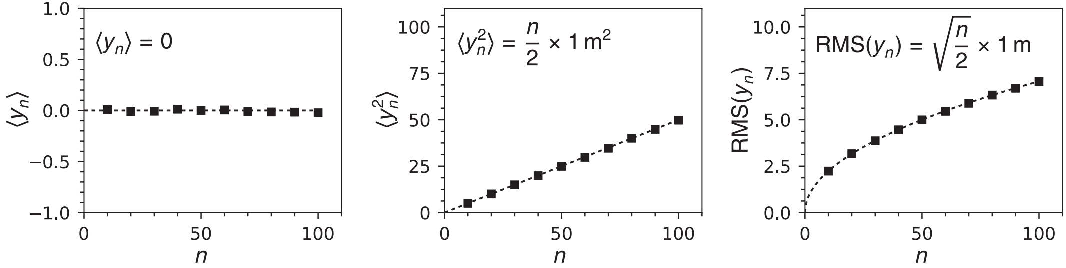
Projections onto the x-axis



- 100,000 simulated random walks for each value of n, from 10 to 100.
- Step length: $\delta = 1\,\mathrm{m}$
- Unbiased random walks: Turns in any direction are equally likely.
- Since $\langle x_n^2 \rangle = n\langle \delta_x^2 \rangle$, it looks as though $\langle \delta_x^2 \rangle = \delta^2/2$

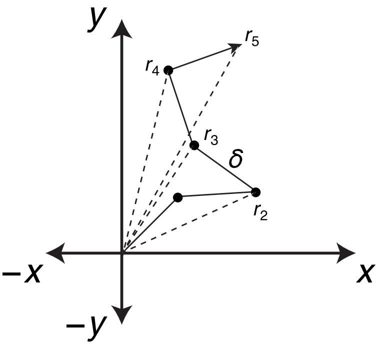
Results from Simulated Two-dimensional Random Walks

Projections onto the y-axis



- 100,000 simulated random walks for each value of n, from 10 to 100.
- Step length: $\delta = 1\,\mathrm{m}$
- It looks as though $\langle y_n^2 \rangle = \langle x_n^2 \rangle$, and $\langle \delta_y^2 \rangle = \langle \delta_x^2 \rangle$, as expected.

Distance from the Starting Point



- r_i is the distance from the starting point to the position after step i.
- r_i is always positive, unlike x_i and y_i
- What are $\langle r_n \rangle$, $\langle r_n^2 \rangle$ and $\mathrm{RMS}(r_n)$?

The Expected Value for r_n^2

- For a single random walk:

  $$r_n^2 = x_n^2 + y_n^2$$

- For two independent random variables, A and B

  $$E(A + B) = E(A) + E(B)$$

- The expected value of r_n^2:

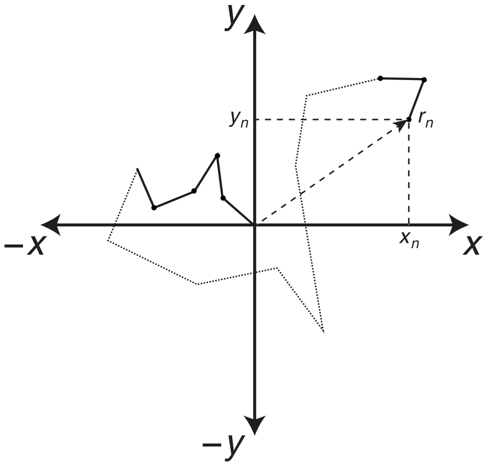
  $$E(r_n^2) = E(x_n^2) + E(y_n^2)$$

  $$\langle r_n^2 \rangle = \langle x_n^2 \rangle + \langle y_n^2 \rangle$$

  $$\langle r_n^2 \rangle = n\langle \delta_x^2 \rangle + n\langle \delta_y^2 \rangle$$

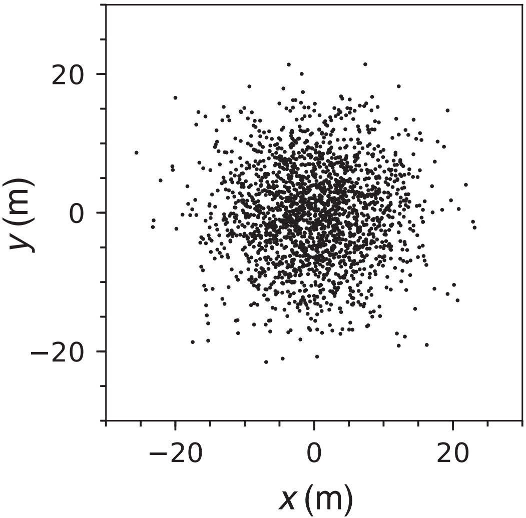
  $$\langle r_n^2 \rangle = 2n\langle \delta_{x,y}^2 \rangle$$

- Are x_n^2 and y_n^2 independent?



Results from Simulated Two-dimensional Random Walks

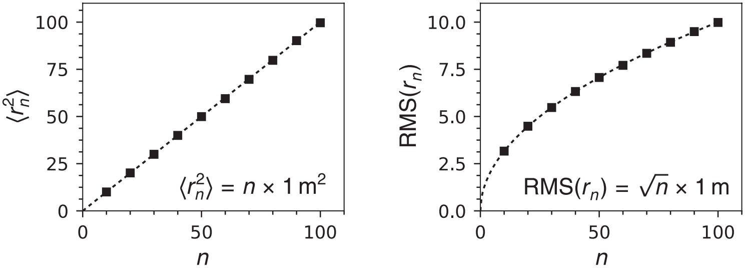
Distribution of endpoints on x-y plane



- 2,000 simulated random walks
- 100 steps
- Unbiased random walks: Turns in any direction are equally likely.
- Step length: $\delta = 1\,\mathrm{m}$
- x_n and y_n aren't completely independent!
- Does $\langle r_n^2 \rangle$ equal $\langle x_n^2 \rangle + \langle y_n^2 \rangle$?

Results from Simulated Two-dimensional Random Walks

End-to-end distance (r)

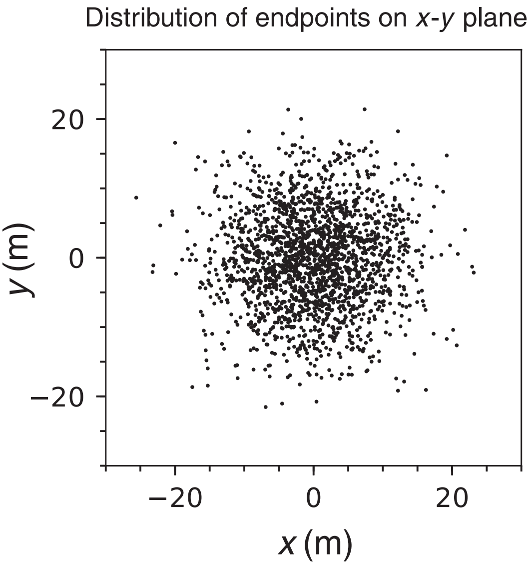


- As predicted if x_n^2 and y_n^2 are independent:

$$\langle r_n^2 \rangle = \langle x_n^2 \rangle + \langle y_n^2 \rangle = n\langle \delta_x^2 \rangle + n\langle \delta_y^2 \rangle$$

$$= 2n\langle \delta_{x,y}^2 \rangle = 2n\left(\frac{1}{2}\delta^2\right) = n\delta^2$$

Results from Simulated Two-dimensional Random Walks



- x_n and y_n are largely independent for the great majority of the random walks.

Major Results So Far for a Two-Dimensional Random Walk

For n steps of length δ:

- Displacement along the x- and y-axes (or any other direction):
  - Mean displacement: $\langle x_n \rangle = \langle y_n \rangle = 0$.
  - Mean-square displacement: $\langle x_n^2 \rangle = \langle y_n^2 \rangle = n\langle \delta_{x,y}^2 \rangle = n\delta^2/2$
  - RMS displacement: $\mathrm{RMS}(x_n) = \mathrm{RMS}(y_n) = \sqrt{(n/2)}\delta$
- Distance from starting point, r:
  - Mean-square displacement: $\langle r_n^2 \rangle = 2n\langle \delta_{x,y}^2 \rangle = n\delta^2$
  - RMS displacement: $\mathrm{RMS}(r_n) = \sqrt{n}\delta$
  - Just like the one-dimensional random walk!
  - Mean displacement: $\langle r \rangle =?$

Clicker Question #1

For a 2-dimensional random walk of a total (fitBit) distance of 100 m and a step length of 5 m, which (if any) of the following are correct?

Choose up to 2.

A) $\mathrm{RMS}(r) \approx 50\,\mathrm{m}$

B) $\langle r^2 \rangle \approx 500\,\mathrm{m}^2$

C) $\langle r^2 \rangle \approx 250\,\mathrm{m}^2$

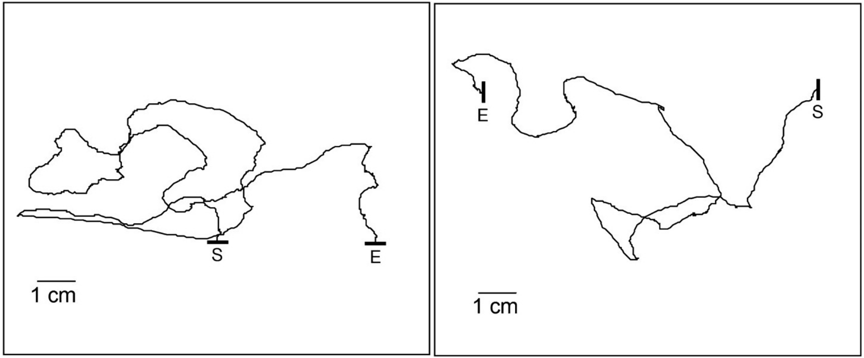
D) $\mathrm{RMS}(r) \approx 22\,\mathrm{m}$

E) None of the above

$$n = 100\,\mathrm{m}/5\,\mathrm{m} = 20$$

$$\langle r^2 \rangle = n\delta^2 = 20 \times (5\,\mathrm{m}^2) = 500\,\mathrm{m}^2$$

Ants on a Walk for Food



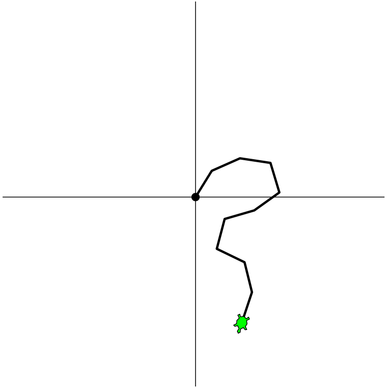
Brachymyrmex depilis (25 s) *Dorymyrmex insanus* (21 s)

- Do either look like a random walk?

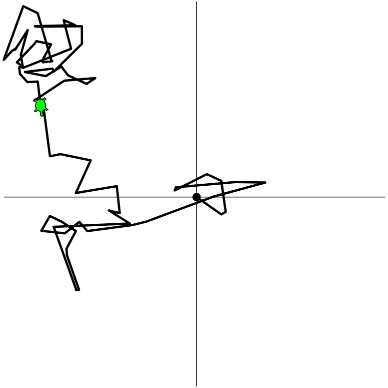
Pearce-Duvet, J. M. C., Elemens, C. P. H. & Feener, D. H. (2011). Walking the line: search behavior and foraging success in ant species. *Behavioral Ecology*, 22, 501–509.
`http://dx.doi.org/10.1093/beheco/arr001`

Simple Variations on the Two-dimensional Random Walk

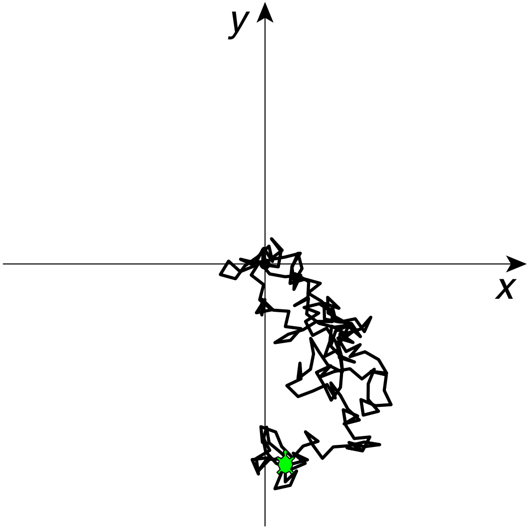
- Constrain change in direction.



- Introduce variation in step length.

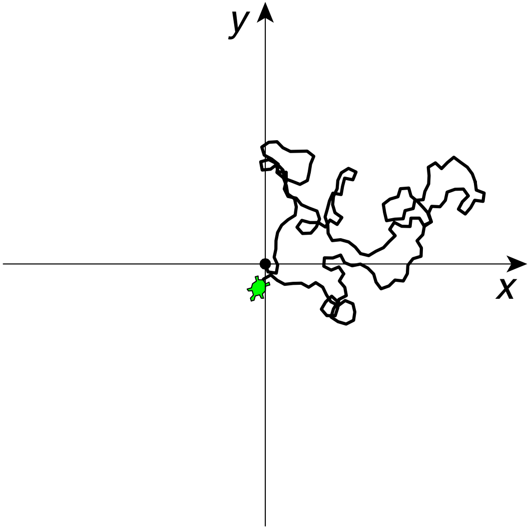


A 'Plain' Random Walk



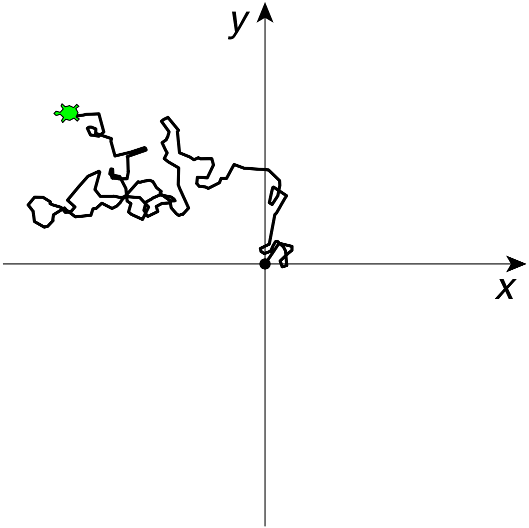
- Step length = 20
- No. steps = 200

A “Correlated” Random Walk



- Turn angle restricted to -90° to 90°
- Step length = 8
- No. steps = 200

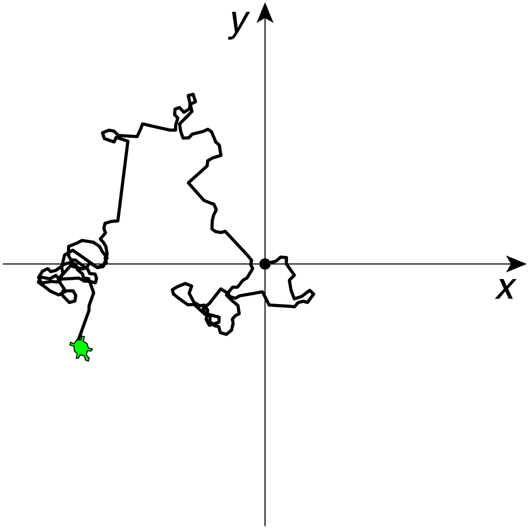
A Random Walk With a Distribution of Step Lengths



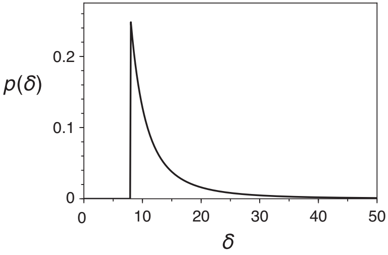
- Turn angle restricted to -90° to 90°
- Half-Gaussian (bell curve) distribution of step lengths
- No. steps = 200

A “Lévy Flight”

A random walk with a “heavy-tailed” distribution of step lengths



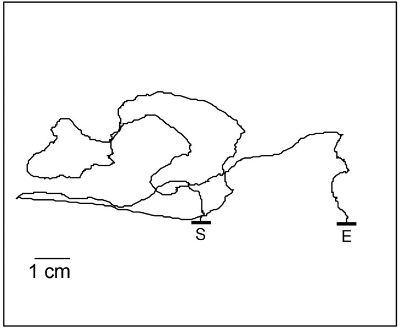
- Turn angle restricted to -90° to 90°
- Pareto distribution of step lengths



- No. steps = 200

Clicker Question #2

What does the ant walk most resemble?



Brachymyrmex depilis
(25 s)

A) A plain random walk

B) A correlated random walk

C) A Lévy Flight

All answers count.