Physical Principles in Biology
Biology 3550
Spring 2024

Lecture 15:

The Gaussian Probability Distribution Function and Introduction to Diffusion

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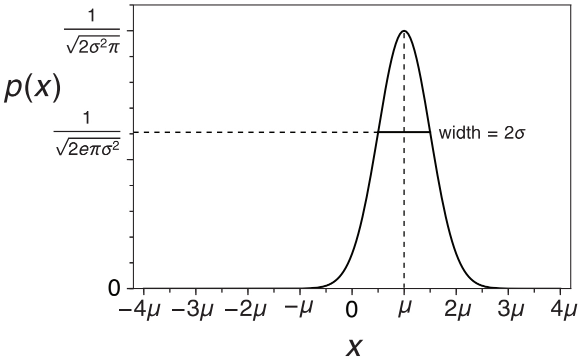
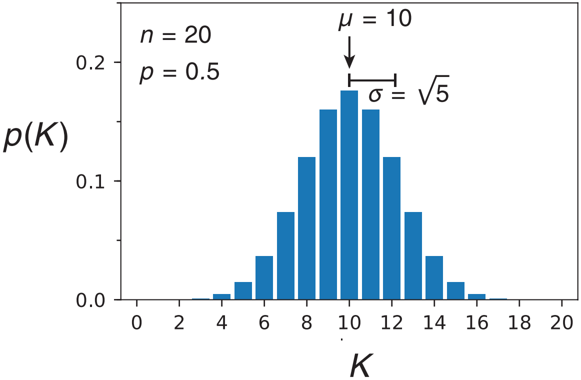
The Gaussian Probability Distribution Function



$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

- Also called the normal probability distribution function.
- Mean $= \mu$
- Variance $= \sigma^2$
- Standard deviation $= \sigma$

Mean and Variance for The Binomial Distribution



- Mean

$$\mu = np_s$$

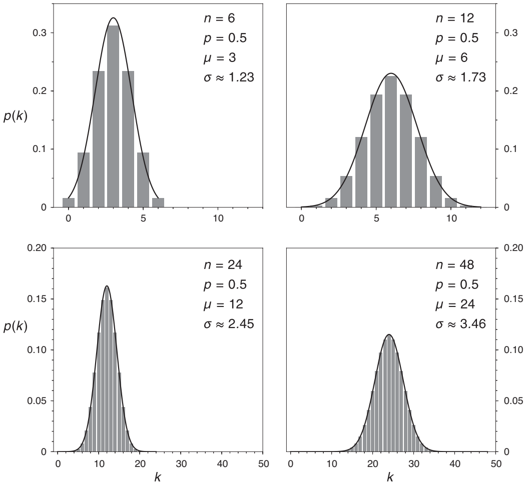
- Variance

$$\sigma^2 = np_s(1 - p_s)$$

- Gaussian probability function to approximate the binomial distribution function with the same mean and variance:

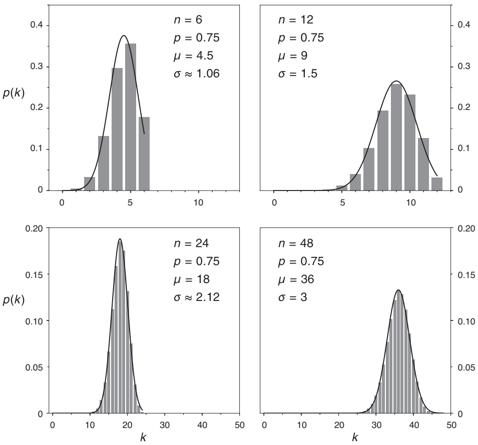
$$p(k) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(k-\mu)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi np_s(1 - p_s)}} e^{-\frac{(k-np_s)^2}{2np_s(1-p_s)}}$$

Approximation of Binomial Distributions by Gaussian Distributions



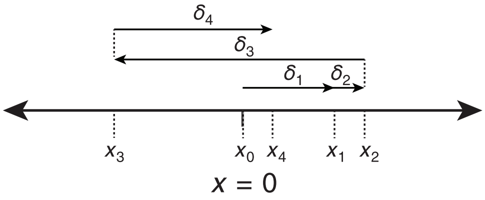
- *n* doesn't have to be very large for a pretty good approximation!

Approximation of Binomial Distributions by Gaussian Distributions



- It doesn't work so well if the binomial distribution is biased, with $p_s \neq 0.5$.
- The Gaussian distribution is always symmetrical, but the binomial distribution only is if $p_s = 0.5$.
- If n is large enough, the Gaussian distribution is a good approximation, even if $p_s \neq 0.5$.

The One-dimensional Random Walk with Continuously Variable Step Length



- Mean displacement:

$$\langle \delta \rangle = E(\delta) = \int_{\delta_{\min}}^{\delta_{\max}} p(\delta)\delta\, d\delta$$

- Mean-squared displacement:

$$\langle \delta^2 \rangle = E(\delta^2) = \int_{\delta_{\min}}^{\delta_{\max}} p(\delta)\delta^2\, d\delta$$

- If $\langle \delta \rangle = 0$:

$$\langle x_n \rangle = 0$$
$$\langle x_n^2 \rangle = n\langle \delta^2 \rangle$$

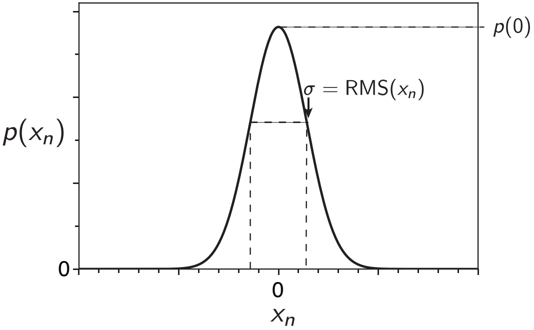
- The mean and variance of x_n:

$$\mu = \int_{x_{n,\min}}^{x_{n,\max}} p(x_n)x_n = \langle x_n \rangle = 0$$

$$\sigma^2 = \int_{x_{n,\min}}^{x_{n,\max}} p(x_n)(x_n - \mu)^2$$

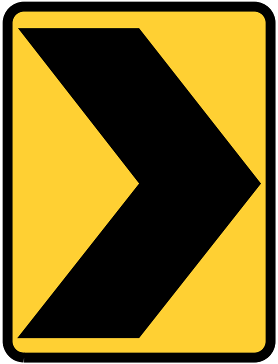
$$= \int_{x_{n,\min}}^{x_{n,\max}} p(x_n)x_n^2 = \langle x_n^2 \rangle$$

The Gaussian Distribution of End Points for a 1-dimensional Random Walk with Variable Displacements



- $\mu = 0$
- $\sigma^2 = \langle x_n^2 \rangle = n\langle \delta^2 \rangle$
- $$p(x_n) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_n-\mu)^2}{2\sigma^2}}$$
  $$= \frac{1}{\sqrt{2\pi\langle x_n^2 \rangle}} e^{-x^2/(2\langle x_n^2 \rangle)}$$
- $$p(0) = \frac{1}{\sqrt{2\pi\langle x_n^2 \rangle}}$$

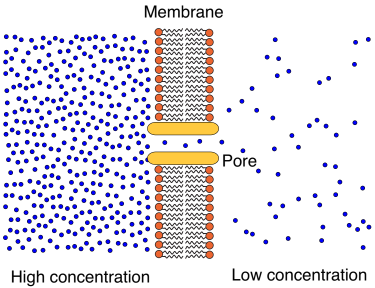
Warning!



Direction Change

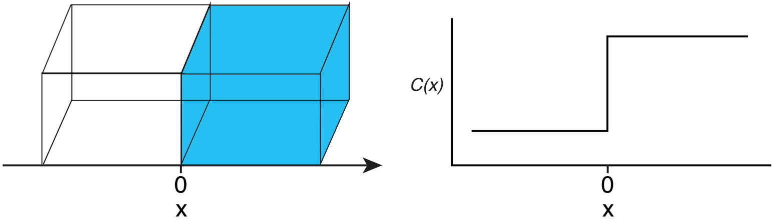
Diffusion

Diffusion of Molecules Across a Cellular Membrane



- How do the concentrations change with time?
- WHY do the concentrations change?
- How fast do the concentrations change?
- How fast do the molecules move?

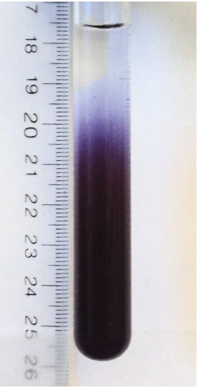
An Idealized Macroscopic Diffusion Experiment



- How will plot of $C(x)$ versus x change with time?
- There's a theory for that! Fick's laws of diffusion

Clicker Question #1

How long has it been since there was a sharp boundary?



A) 10 min

B) 1 hr

C) 12 hr

D) 48 hr

All answers count! (for now)

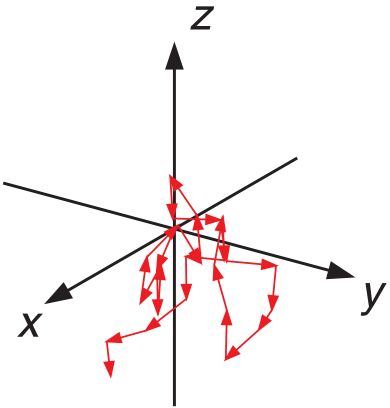
Adolf Eugen Fick

- 1829-1901
- Physician and physiologist
- Nephew, Adolf Gaston Eugen Fick, invented contact lenses in 1888.

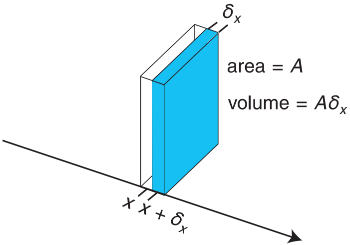
https://en.wikipedia.org/wiki/Adolf_Eugen_Fick Portrait by Anton Klamroth.

Diffusion as a Random Walk

- $\langle l_i^2 \rangle$: Mean-square step length in three dimensions
- $\delta_x = \sqrt{\langle l_i^2 \rangle / 3}$: RMS displacement along the x-direction.
- τ: Average time interval between changes in direction
- t: Time interval of interest
- $n = t/\tau$: Average number of steps in time t
- $\langle x_n^2 \rangle = n\delta_x^2 = t\delta_x^2/\tau$

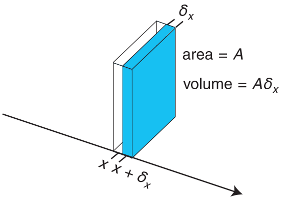


Diffusion Across a Thin Slice of Volume



- Slices are δ_x thick and have cross-sectional area across the x-axis of A
- Volume of each slice is $A\delta_x$
- During time τ, all of the molecules will move (on average) the distance δ_x along the x-axis, to the left or right.
- In a given slice, 1/2 of the molecules will move to the right and half to the left.

Diffusion Across a Thin Slice of Volume

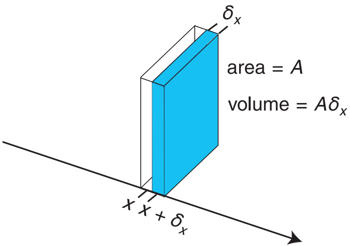


- $N_x =$ number of molecules starting in the slice centered at x.
- $N_{x+\delta_x} =$ number of molecules starting in the slice centered at $x + \delta_x$
- The net number of molecules moving from slice x to slice $x + \delta_x$, in time τ:

$$dN = \frac{1}{2} N_x - \frac{1}{2} N_{x+\delta_x}$$

$$= -\frac{1}{2}\left(N_{x+\delta_x} - N_x\right)$$

Diffusion Across a Thin Slice of Volume



- Definition: Flux, $J =$ net number of molecules (or moles) moving past a cross section, per unit time, per unit area.

$$J = -\frac{1}{A\tau}\frac{1}{2}\left(N_{x+\delta_x} - N_x\right)$$

Movement is defined in the direction of positive x.

Diffusion Across a Thin Slice of Volume

- Express number of molecules in each slice in terms of the concentrations and volumes of each slice.

$$N_x = C_x \cdot V = C_x A \delta_x$$

$$N_{x+\delta_x} = C_{x+\delta_x} \cdot V = C_{x+\delta_x} A \delta_x$$

- Re-write the flux equation as:

$$\begin{aligned} J &= -\frac{1}{A\tau}\frac{1}{2}\left(N_{x+\delta_x} - N_x\right) \\ &= -\frac{1}{A\tau}\frac{1}{2}\left(C_{x+\delta_x} A\delta_x - C_x A\delta_x\right) \\ &= -\frac{\delta_x}{\tau}\frac{1}{2}\left(C_{x+\delta_x} - C_x\right) \end{aligned}$$

Diffusion Across a Thin Slice of Volume

- Write the concentration difference in terms of a derivative with respect to x:

$$\frac{dC}{dx} = \lim_{\delta_x \to 0} \frac{\left(C_{x+\delta_x} - C_x\right)}{\delta_x}$$

in the limit of small δ_x:

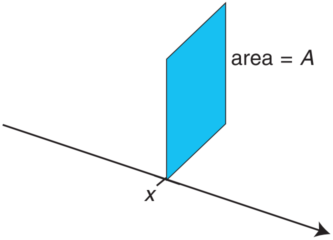
$$\left(C_{x+\delta_x} - C_x\right) = \delta_x \frac{dC}{dx}$$

- Flux equation:

$$J = -\frac{\delta_x}{\tau}\frac{1}{2}\left(C_{x+\delta_x} - C_x\right) = -\frac{\delta_x^2}{2\tau}\frac{dC}{dx}$$

Fick's First Law of Diffusion

$$J = -\frac{\delta_x^2}{2\tau}\frac{dC}{dx}$$



- Symbols:
  - $J =$ flux of molecules per unit area per unit time:
  - $\delta_x =$ RMS step length along the x-direction.
  - $\tau =$ average duration of random steps.
  - $\frac{dC}{dx}$ = derivative of concentration with x, "concentration gradient."
- If concentration increases with x, flux is in the negative x direction.
- Why do molecules "move down the concentration gradient"?

The Diffusion Coefficient, D

- Consider the term from Fick's first law: $\frac{\delta_x^2}{2\tau}$

  Both δ_x and τ are parameters describing the random walk of molecules (or larger particles) undergoing diffusion, and are constant for a given type of particle under defined solution conditions.

- Define a new parameter, the diffusion coefficient, D:

$$D = \frac{\delta_x^2}{2\tau}$$

- The usual form of Fick's first law:

$$J = -D\frac{dC}{dx}$$

- D can be experimentally determined, without knowing anything about the microscopic random walk steps.