Physical Principles in Biology
Biology 3550
Fall 2018

Lecture 16:

Diffusion: Fick's First Law

Friday, 28 September 2018

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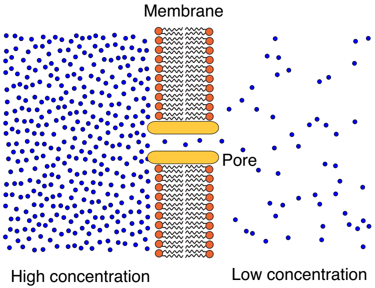
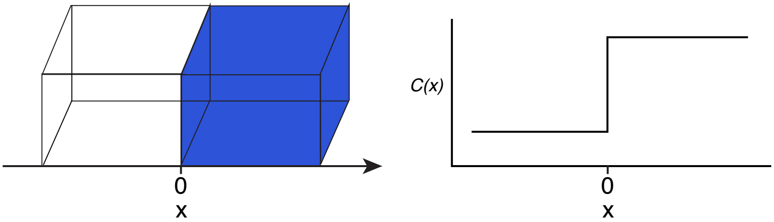
Diffusion of Molecules Across a Cellular Membrane



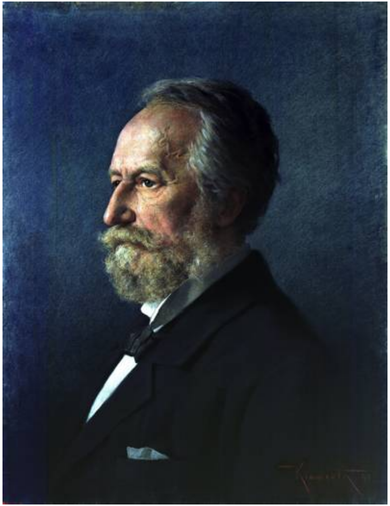
- How do the concentrations change with time?
- WHY do the concentrations change?
- How fast do the concentrations change?
- How fast do the molecules move?

An Idealized Macroscopic Diffusion Experiment



- How will plot of $C(x)$ versus x change with time?
- There's a theory for that! Fick's laws of diffusion

Adolf Eugen Fick

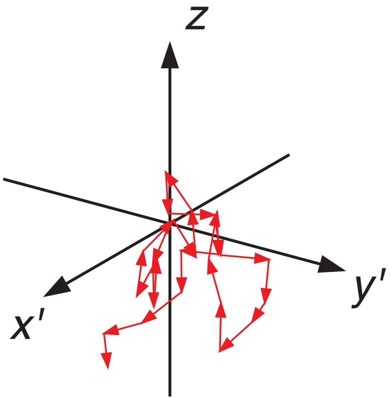


- 1829-1901
- Physician and physiologist
- Nephew, Adolf Gaston Eugen Fick, invented contact lenses in 1888.

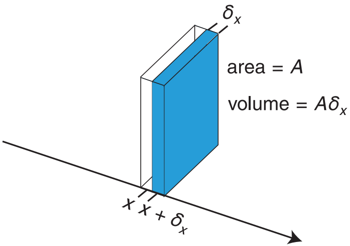
https://en.wikipedia.org/wiki/Adolf_Eugen_Fick Portrait by Anton Klamroth.

Diffusion as a Random Walk

- $\langle\delta^2\rangle$: Mean-square step length in three dimensions
- $\delta_x = \sqrt{\langle\delta^2\rangle/3}$: RMS displacement along the x-direction.
- τ: Average time interval between changes in direction
- t: Time interval of interest
- $n = t/\tau$: Average number of steps in time t
- $\langle x_n^2\rangle = n\delta_x^2 = t\delta_x^2/\tau$

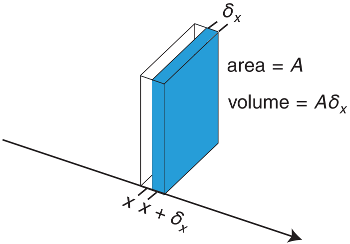


Diffusion Across a Thin Slice of Volume



- Slices are δ_x thick and have cross-sectional area across the x-axis of A
- Volume of each slice is $A\delta_x$
- During time τ, all of the molecules will move (on average) the distance δ_x.
- In a given slice, 1/2 of the molecules will move to the right and half to the left.

Diffusion Across a Thin Slice of Volume

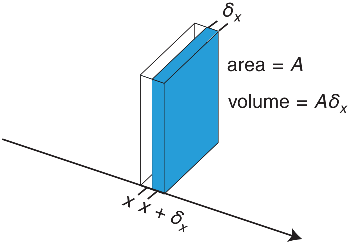


- $N_x =$ number of molecules in the slice centered at x.
- $N_{x+\delta_x} =$ number of molecules in the slice centered at $x + \delta_x$
- The net number of molecules moving from slice x to slice $x + \delta_x$:

$$\delta N = \frac{1}{2} N_x - \frac{1}{2} N_{x+\delta_x}$$

$$= -\frac{1}{2}\left(N_{x+\delta_x} - N_x\right)$$

Diffusion Across a Thin Slice of Volume



- Definition: Flux, $J =$ net number of molecules (or moles) moving past a cross section, per unit time, per unit area.

$$J = -\frac{1}{A\tau}\frac{1}{2}\left(N_{x+\delta_x} - N_x\right)$$

Movement is defined in the direction of positive x.

Diffusion Across a Thin Slice of Volume

- Express number of molecules in each slice in terms of the concentrations and volumes of each slice.

$$N_x = C_x \cdot V = C_x A \delta_x$$

$$N_{x+\delta_x} = C_{x+\delta_x} \cdot V = C_{x+\delta_x} A \delta_x$$

- Re-write the flux equation as:

$$\begin{aligned} J &= -\frac{1}{A\tau}\frac{1}{2}\left(N_{x+\delta_x} - N_x\right) \\ &= -\frac{1}{A\tau}\frac{1}{2}\left(C_{x+\delta_x} A\delta_x - C_x A\delta_x\right) \\ &= -\frac{\delta_x}{\tau}\frac{1}{2}\left(C_{x+\delta_x} - C_x\right) \end{aligned}$$

Diffusion Across a Thin Slice of Volume

- Write the concentration difference in terms of a derivative with respect to x:

$$\frac{dC}{dx} = \lim_{\delta_x \to 0} \frac{\left(C_{x+\delta_x} - C_x\right)}{\delta_x}$$

in the limit of small δ_x:

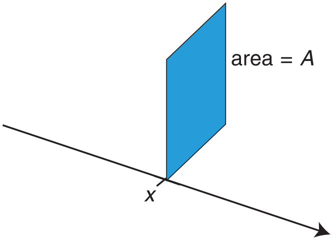
$$\left(C_{x+\delta_x} - C_x\right) = \delta_x \frac{dC}{dx}$$

- Flux equation:

$$J = -\frac{\delta_x}{\tau}\frac{1}{2}\left(C_{x+\delta_x} - C_x\right) = -\frac{\delta_x^2}{2\tau}\frac{dC}{dx}$$

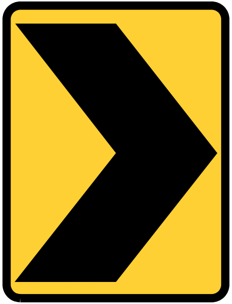
Fick's First Law of Diffusion

$$J = -\frac{\delta_x^2}{2\tau}\frac{dC}{dx}$$



- Symbols:
  - $J =$ flux of molecules per unit area per unit time:
  - $\delta_x =$ RMS displacement along the x-direction.
  - $\tau =$ average duration of random steps.
  - $\frac{dC}{dx}$ = derivative of concentration with x, "concentration gradient."
- If concentration increases with x, flux is in the negative x direction.

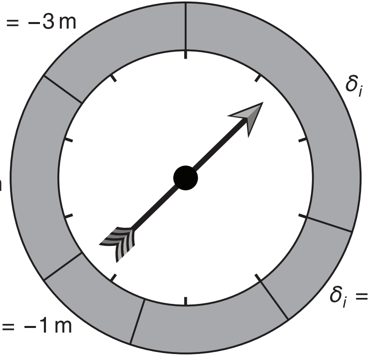
Warning!



Direction Change

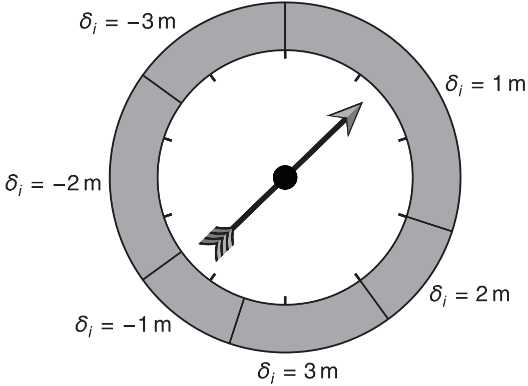
Group-work Problems

A Spinner for One-dimensional Random Walks



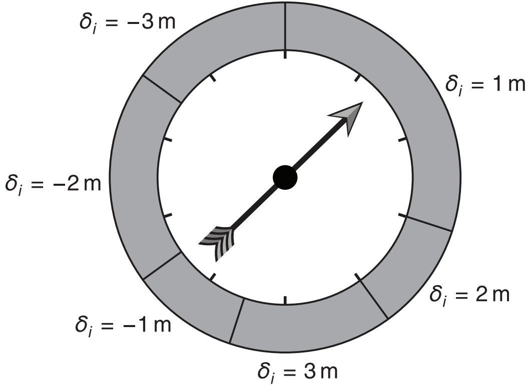
- δ_i is the displacement along the x-axis for an individual random walk step and can take on values of 1, 2, 3, -1, -2 or -3 m
- The ticks on the inner part of the spinner divide the scale up into 10 equal parts.
- Calculate the average x-displacement, $\langle \delta_i \rangle$.

A Spinner for One-dimensional Random Walks



- δ_i is the displacement along the x-axis for an individual random walk step and can take on values of 1, 2, 3, -1, -2 or -3 m
- The ticks on the inner part of the spinner divide the scale up into 10 equal parts.
- Calculate the mean-square x-displacement, $\langle \delta_i^2 \rangle$.

A Spinner for One-dimensional Random Walks



- For a large number of one-dimensional random walks of 150 steps each, using this spinner and starting at $x = 0$, calculate:
  a. The average position at the ends of the random walks, $\langle x_n \rangle$.
  b. The mean-square average position at the ends of the random walks, $\langle x_n^2 \rangle$.
  c. The root-mean-square distance from the start of the random walks to the ends.