Physical Principles in Biology
Biology 3550
Spring 2024

Lecture 16:

Diffusion Continued: Fick's First and Second Laws

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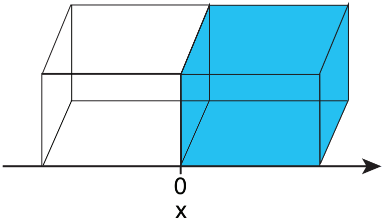
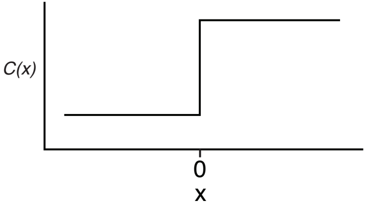
University of Utah
goldenberg@biology.utah.edu

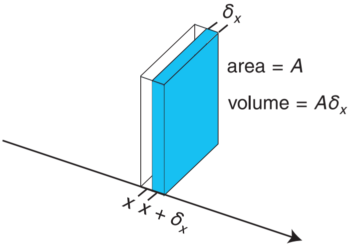
An Idealized Macroscopic Diffusion Experiment





- How will plot of $C(x)$ versus x change with time?

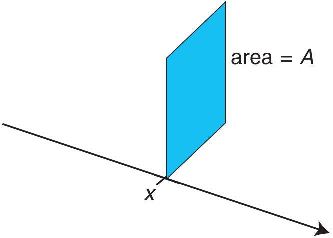
Diffusion Across a Thin Slice of Volume



- Slices are δ_x thick and have cross-sectional area across the x-axis of A
- Volume of each slice is $A\delta_x$
- During time τ, all of the molecules will move (on average) the distance δ_x along the x-axis.
- In a given slice, 1/2 of the molecules will move to the right and half to the left.

Fick's First Law of Diffusion

$$J = -\frac{\delta_x^2}{2\tau}\frac{dC}{dx}$$



- Symbols:
  - $J =$ flux of molecules per unit area per unit time:
  - $\delta_x =$ RMS step length along the x-direction.
  - $\tau =$ average duration of random steps.
  - $\frac{dC}{dx}$ = derivative of concentration with x, "concentration gradient."
- If concentration increases with x, flux is in the negative x direction.

The Diffusion Coefficient, D

- Consider the term from Fick's first law: $\frac{\delta_x^2}{2\tau}$

  Both δ_x and τ are parameters describing the random walk of molecules (or larger particles) undergoing diffusion, and are constant for a given type of particle under defined solution conditions.

- Define a new parameter, the diffusion coefficient, D:

$$D = \frac{\delta_x^2}{2\tau}$$

- The usual form of Fick's first law:

$$J = -D\frac{dC}{dx}$$

- D can be experimentally determined, without knowing anything about the microscopic random walk steps.

Clicker Question #1

What are the units of the diffusion coefficient: $D = \frac{\delta_x^2}{2\tau}$?

A) mol/s

B) m^2/s

C) mol/m/s

D) m/s

Clicker Question #2

What are the units of the concentration gradient: $\frac{dC}{dx}$?

A) $\mathrm{mol/m}$

B) $\mathrm{mol/m^2}$

C) $\mathrm{mol/m^3}$

D) $\mathrm{mol/m^4}$

Dimensional Analysis of Fick's First Law

- The diffusion coefficient: $D = \frac{\delta_x^2}{2\tau}$, units: m^2/s
- The concentration gradient: $\frac{dC}{dx}$

  C (concentration) units: $\mathrm{moles}/\mathrm{m}^3$
  x (distance) units: m

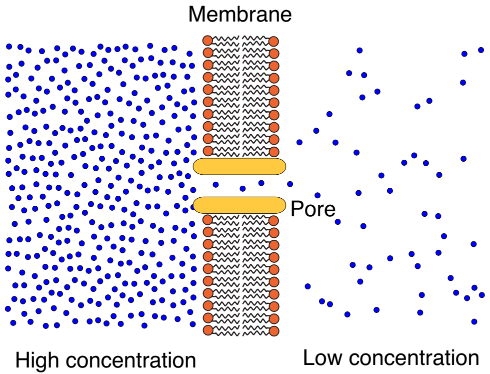
  $\frac{dC}{dx}$ has units of $\frac{\mathrm{moles}/\mathrm{m}^3}{\mathrm{m}} = \frac{\mathrm{moles}}{\mathrm{m}^4}$

- Flux, J

$$J = -D\frac{dC}{dx} = \frac{\mathrm{m}^2}{\mathrm{s}}\frac{\mathrm{moles}}{\mathrm{m}^4} = \frac{\mathrm{moles}}{\mathrm{s}\cdot\mathrm{m}^2}$$

moles per unit time per unit area, as originally defined.

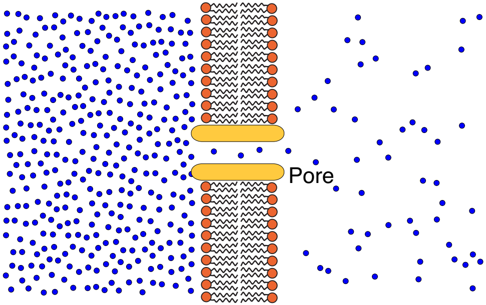
Diffusion of Molecules Across a Cellular Membrane



Some plausible values:

- Diffusion coefficient for a small molecule: $\approx 10^{-10}\,\mathrm{m}^2/\mathrm{s}$ (to be considered later)
- Thickness of a biological membrane: $\approx 3\,\mathrm{nm}$
- Diameter of a molecular pore: $\approx 1\,\mathrm{nm}$
- Concentrations: 50 mM and 5 mM

Clicker Question #3



- Thickness of membrane: $\approx 3\,\text{nm}$
- Diameter of pore: $\approx 1\,\text{nm}$
- Concentrations: 50 mM and 5 mM

Estimate the concentration gradient:

$$\frac{dC}{dx} \approx \frac{\Delta C}{\Delta x}$$

A) $1.5 \times 10^{3}\,\text{moles}/\text{m}^{4}$

B) $1.5 \times 10^{6}\,\text{moles}/\text{m}^{4}$

C) $1.5 \times 10^{9}\,\text{moles}/\text{m}^{4}$

D) $1.5 \times 10^{10}\,\text{moles}/\text{m}^{4}$

E) $4.5 \times 10^{10}\,\text{moles}/\text{m}^{4}$

Applying Fick's First Law

- Approximate the concentration gradient:

$$\frac{dC}{dx} \approx \frac{\text{concentration difference}}{\text{membrane thickness}} = \frac{50\,\text{mM} - 5\,\text{mM}}{3\,\text{nm}} = 15\,\text{mM}/\text{nm}$$

- Convert this to units with moles and meters:

$$15\,\text{mM}/\text{nm} = \frac{0.015\,\text{moles}/\text{L}}{10^{-9}\,\text{m}} \times \frac{10^3\,\text{L}}{\text{m}^3} = 1.5 \times 10^{10}\,\text{moles}/\text{m}^4$$

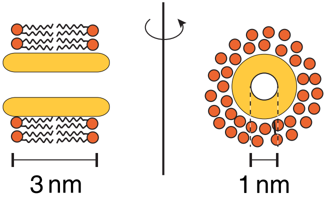
- Flux from Fick's first law:

$$J = -D\frac{dC}{dx} = -10^{-10}\,\text{m}^2/\text{s} \times 1.5 \times 10^{10}\,\text{moles}/\text{m}^4 = -1.5\,\text{moles}/(\text{s} \cdot \text{m}^2)$$

Negative sign indicates that diffusion is in the direction opposite to the concentration gradient.

Diffusion Through a Membrane Pore

- The pore:



  Diameter = 1 nm

  Area = $\pi r^2 = \pi(0.5\,\text{nm})^2 = 7.8 \times 10^{-19}\,\text{m}^2$

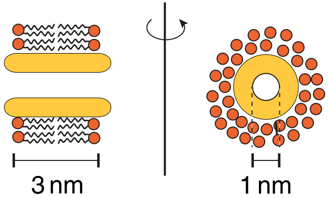
- Flux:

$$J = -1.5\,\text{moles}/(\text{s}\cdot\text{m}^2)$$

- Moles per second through the pore:

$$1.5\,\text{moles}/(\text{s}\cdot\text{m}^2) \times 7.8 \times 10^{-19}\,\text{m}^2 = 1.2 \times 10^{-18}\,\text{moles}/\text{s}$$

Diffusion Through a Membrane Pore



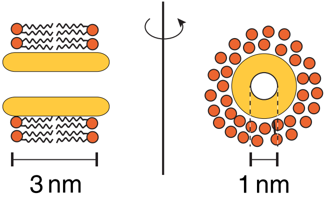
- Moles per second through the pore:

$$1.5\,\text{moles}/(\text{s}\cdot\text{m}^2)\times 7.8\times 10^{-19}\,\text{m}^2 = 1.2\times 10^{-18}\,\text{moles}/\text{s}$$

- Molecules per second through the pore:

$$1.2\times 10^{-18}\,\text{moles}/\text{s}\times 6.02\times 10^{23}\,\text{molecules}/\text{mole} = 7\times 10^{5}\,\text{molecules}/\text{s}$$

How Many Molecules are in the Pore?



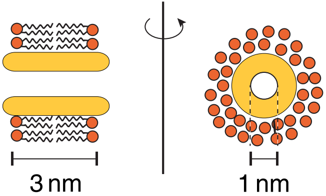
- Pore volume:

$$V = \pi r^2 \times l = \pi (0.5\,\text{nm})^2 \times 3\,\text{nm}$$
$$= 2.4\,\text{nm}^3 \times (10^{-9}\,\text{m/nm})^3 = 2.4 \times 10^{-27}\,\text{m}^3$$
$$= 2.4 \times 10^{-27}\,\text{m}^3 \times 10^3\,\text{L/m}^3 = 2.4 \times 10^{-24}\,\text{L}$$

- Assume that the concentration in the pore is the mean of the concentrations on the two sides:

$$C = (50\,\text{mM} - 5\,\text{mM})/2 = 22\,\text{mM} = 0.022\,\text{moles/L}$$

How Many Molecules are in the Pore?



- Concentration: 0.022 moles/L
- Moles in pore:

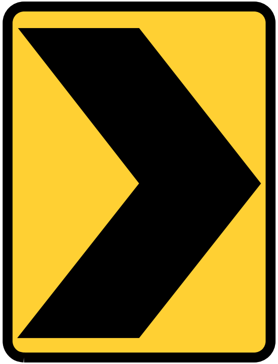
$$0.022\,\text{moles}/\text{L} \times 2.4 \times 10^{-24}\,\text{L} = 5.3 \times 10^{-26}\,\text{moles}$$

- Molecules in pore:

$$5.3 \times 10^{-26}\,\text{moles} \times 6.02 \times 10^{23}\,\text{molecules}/\text{mole} \approx 0.03\,\text{molecules}$$

Most of the time, the pore is "empty"!

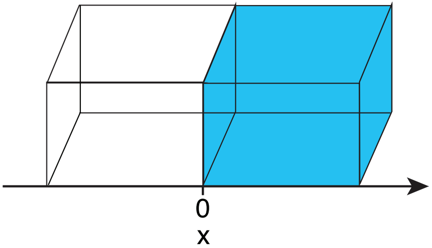
Warning!

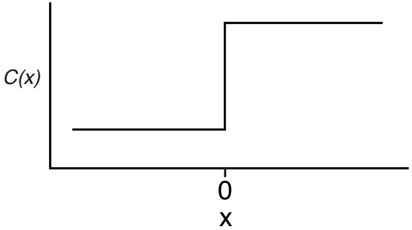


Direction Change

Change in Concentration with Time

An Idealized Macroscopic Diffusion Experiment

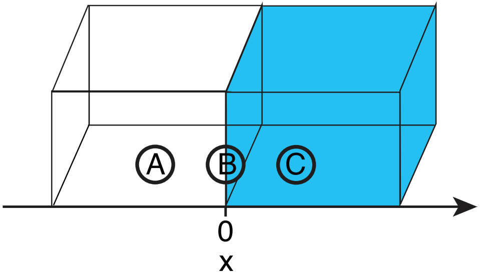


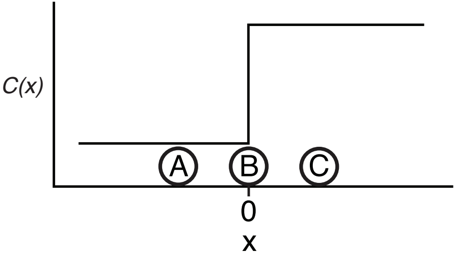


- How will plot of $C(x)$ versus x change with time?

Clicker Question #4

Where will the flux, *J*, be greatest?



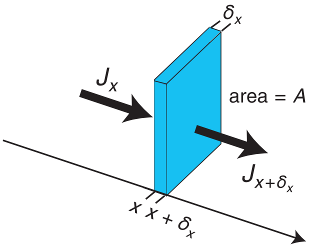


- At point B, where the concentration gradient is greatest.
- But, the molecules move at the same rate everywhere!

Fick's Second Law of Diffusion

- How does the concentration at a given point change with time?
- Net number of molecules moving to the right at two sides of a slice, during interval dt:

$$N_x = J_x A dt$$
$$N_{x+\delta_x} = J_{x+\delta_x} A dt$$



- Change in number of molecules in the slice:

$$\begin{aligned} dN &= N_x - N_{x+\delta_x} \\ &= AJ_x dt - AJ_{x+\delta_x} dt \\ &= Adt\left(J_x - J_{x+\delta_x}\right) \end{aligned}$$

Fick's Second Law of Diffusion

- Change in concentration in the slice:

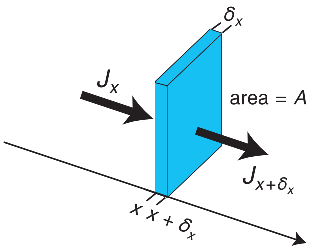
$$dC = \frac{dN}{A\delta_x} = \frac{A\,dt\left(J_x - J_{x+\delta_x}\right)}{A\delta_x}$$

$$= -dt\frac{J_{x+\delta_x} - J_x}{\delta_x}$$

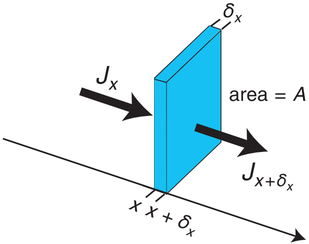
- In the limit of small dt and small δ_x:

$$\frac{dC}{dt} = -\frac{J_{x+\delta_x} - J_x}{\delta_x} = -\frac{dJ}{dx}$$

- How does J change with x?



Fick's Second Law of Diffusion



- Fick's first law:

$$J = -D\frac{dC}{dx}$$

- Derivative of J with respect to x:

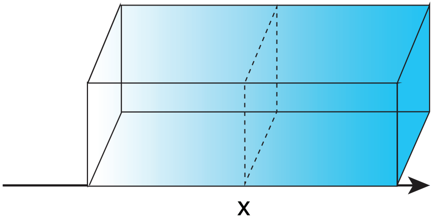
$$\frac{dJ}{dx} = -D\frac{d^2C}{dx^2}$$

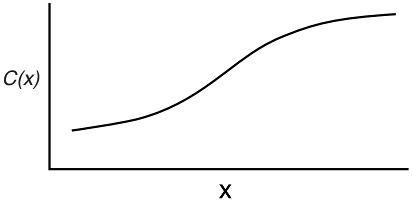
- Fick's second law:

$$\frac{dC}{dt} = -\frac{dJ}{dx} = D\frac{d^2C}{dx^2}$$

Also called the diffusion equation.
What is it good for? How do we use it?

Fick's First and Second Laws of Diffusion





- First law:

$$J = -D\frac{dC}{dx}$$

- Flux, J, at position x is proportional to the concentration gradient at that position.

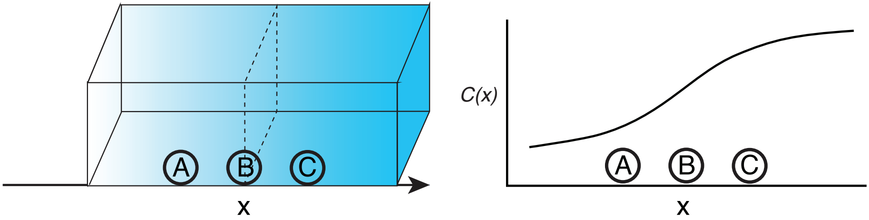
- Second law:

$$\frac{dC}{dt} = D\frac{d^2C}{dx^2}$$

- Rate of change in concentration at position x is proportional to the derivative of the concentration gradient.

Clicker Question #5

Where will the concentration increase most rapidly?



At point A, where the concentration gradient increases most rapidly with respect to x.

Fick's Second Law of Diffusion

$$\frac{dC}{dt} = D\frac{d^2C}{dx^2}$$

- A "second-order differential equation".
- The solution to the equation is a function:

$$C = f(x, t)$$

  that satisfies the equation:

$$\frac{df(x,t)}{dt} = D\frac{d^2f(x,t)}{dx^2}$$

- The trick is to find $C = f(x, t)$.
- The solution depends on the shape of the volume and the initial concentrations, the *boundary conditions*.