Physical Principles in Biology
Biology 3550
Spring 2024

Lecture 5:

Probability: The Basic Rules

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Announcements

- Problem set 1:
  - Due 11:59 PM, Tuesday, 23 January.
  - Download problems from Canvas.
  - Upload work to Gradescope.
  - Work must be typed!
- Quiz 1:
  - Friday, 26 January
  - 25 min, second half of class.

Probability: Some Definitions

- Outcomes – Possible results of a probabilistic process
  - For a coin toss: coin lands heads-up (H) or tails-up (T)
  - For a roll of a six-sided die: The number of spots on the side that lands up (1, 2, 3, 4, 5 and 6)
  - Distinguished from “events”, to be defined shortly
- Probability – A number with a possible value from 0 to 1, associated with a single outcome.
  - $p = 0$: Outcome will never occur.
  - $p = 1$: Outcome will always occur.
  - The sum of the probabilities of all possible outcomes of an experiment must equal 1.
  - What does a probability greater than zero but less than one mean?

The Sample Space, S

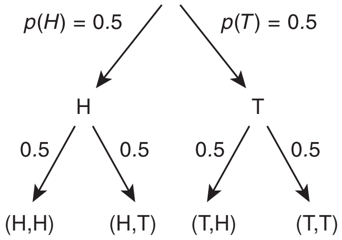
- A set of possible outcomes that satisfies these requirements:
  - The sample set must be complete: It must include all possible results.
  - The elements in the sample set must not overlap.
  - The sum of the probabilities for all of the elements in the sample set must equal 1.
- Often, there are multiple possible ways to define the sample space for a probabilistic process.
- Some examples:
  - For a single coin toss:
    $S = \{T, H\}$
    Curly braces are used to indicate sets.
  - For two independent coin tosses:
    $S = \{(H, H), (H, T), (T, H), (T, T)\}$
    Ordered pairs representing the results of the two tosses.

Events

- An event is a subset of the sample space.
- Some possible events defined for two coin tosses:
  - Two heads: $2H = \{(H, H)\}$
  - Two tails: $2T = \{(T, T)\}$
  - One heads and one tails: $1H1T = \{(H, T), (T, H)\}$
- The outcomes defined in the sample space are events, but additional events can usually be defined by grouping outcomes together.
- Some other events that can be defined for two coin tosses:
  - One or more heads: $1^{+}H = \{(H, H), (H, T), (T, H)\}$
  - One or more tails: $1^{+}T = \{(H, T), (T, H), (T, T)\}$

Calculating Probabilities: Sequential Trials

- Two coin tosses:



- Probabilities are multiplied

$$p\big((H,H)\big) = p(H)p(H) = 0.5 \times 0.5 = 0.25$$
$$p\big((H,T)\big) = p(H)p(T) = 0.5 \times 0.5 = 0.25$$
$$p\big((T,H)\big) = p(T)p(H) = 0.5 \times 0.5 = 0.25$$
$$p\big((T,T)\big) = p(T)p(T) = 0.5 \times 0.5 = 0.25$$

- Multiplication of probabilities is usually associated with “and”.

Calculating Probabilities: Groups of Outcomes or Events

- An event defined earlier for two coin tosses:

  One heads and tails: $1H1T = \{(H, T), (T, H)\}$

- Probability is calculated as a sum:

$$
\begin{aligned}
p(1H1T) &= p\big((H, T)\big) + p\big((T, H)\big) \\
&= p(H)p(T) + p(T)p(H) \\
&= 0.5 \times 0.5 + 0.5 \times 0.5 \\
&= 0.25 + 0.25 = 0.5 \qquad \text{(for a fair coin)}
\end{aligned}
$$

- Addition of probabilities is usually associated with “or”.

Another Example

- An event defined for two coin tosses:

  One or more heads: $1^+H = \{(H,H),(H,T),(T,H)\}$

- Calculation of probability:

$$\begin{aligned} p(1^+H) &= p\big((H,H)\big) + p\big((H,T)\big) + p\big((H,T)\big) \\ &= 0.25 + 0.25 + 0.25 = 0.75 \qquad \text{(for a fair coin)} \end{aligned}$$

- Another way:
  - 1^+H includes all of the sample set, except (T,T).
  - For the entire sample set, the sum of probabilities is 1.

$$\begin{aligned} p(1^+H) &= 1 - p\big((T,T)\big) \\ &= 1 - 0.25 = 0.75 \qquad \text{(for a fair coin)} \end{aligned}$$

Group Problem #1

Suppose that you and a friend are playing a game in which one of you tosses a coin three times in succession, and the result of the game is defined in terms of the number of times the coin lands heads-up.

1. Define the sample set for one round of the game, *i.e.,* all of the possible outcomes defined in terms of the sequence of heads (H) and tails (T).
2. Assuming that the coin is fair, calculate the probability of each of the outcomes in the sample set, assuming that the coin is fair.
3. Define a set of four events, E_0, E_1, E_2 and E_3, where the subscript represents the number of heads. Define each of these events in terms of the elements of the sample set, and calculate the probability of each of these events.

Group Problem #1

1. The sample set:

$$S = \{(H,H,H),(H,H,T),(H,T,H),(H,T,T),\\ (T,H,H),(T,H,T),(T,T,H),(T,T,T)\}$$

2. The probability of each outcome:

$$p = \left(\frac{1}{2}\right)^3 = \frac{1}{2^3} = \frac{1}{8}$$

3. Events:
   - Event E_0, number of heads $= 0$:

$$E_0 = \{(T,T,T)\}$$
$$p(E_0) = \frac{1}{8}$$

Group Problem #1

3. Event (contd.):

- Event E_1, number of heads $= 1$:

  $$E_1 = \{(H, T, T), (T, H, T), (T, T, H)\}$$

  $$p(E_1) = \frac{3}{8}$$

- Event E_2, number of heads $= 2$:

  $$E_2 = \{(H, H, T), (H, T, H), (T, H, H)\}$$

  $$p(E_2) = \frac{3}{8}$$

- Event E_2, number of heads $= 3$:

  $$E_3 = \{(H, H, H)\}$$

  $$p(E_3) = \frac{1}{8}$$

Group Problem #2

- A coin has been tossed 10 times and has landed heads-up each time.
- What is the probability that it will land heads-up the next time?

Group Problem #3

- Assume that:
  - The King has one sibling.
  - The King is male, and his sibling is a brother or sister.
  - The crown is inherited as it is in Britain (males first).
- What is the probability that the King's sibling is a sister?

The King and His Sibling

- The sample set for two siblings:

$$S = \{(M, M), (M, F), (F, M), (F, F)\}$$

M: Male, F: Female

- But, we know that one of the siblings is a male (the King).
- With this information, the sample set is reduced to:

$$S = \{(M, M), (M, F), (F, M)\}$$

- Two of the three outcomes include a sister.
- The probability that the King's sibling is a sister is 2/3!