Physical Principles in Biology
Biology 3550
Fall 2018

Lecture 8:

Plinko Probabilities, Part II

Binomial Coefficients and the Binomial Distribution Function

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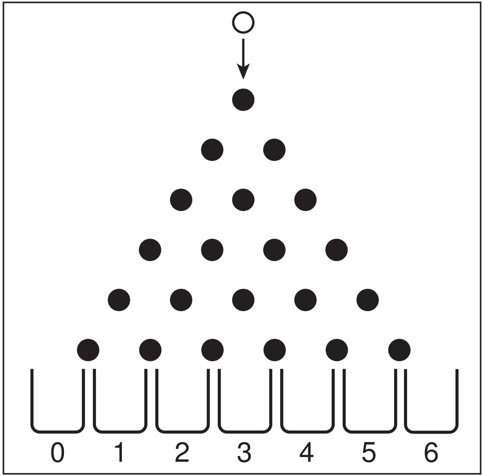
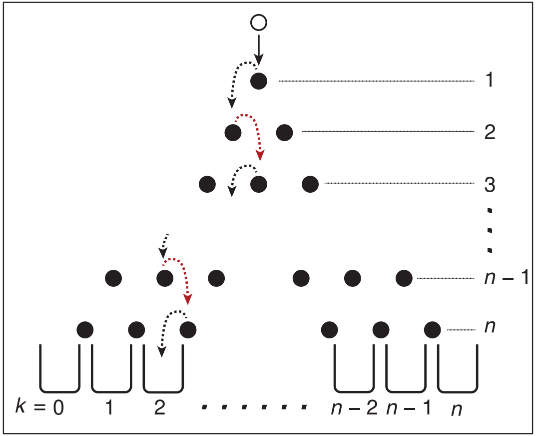
University of Utah
goldenberg@biology.utah.edu

Probabilities for the Six-row Plinko



Bucket No.	Paths	Probability
0	1	$1/64 \approx 0.016$
1	6	$6/64 \approx 0.094$
2	15	$15/64 \approx 0.234$
3	20	$20/64 \approx 0.312$
4	15	$15/64 \approx 0.234$
5	6	$6/64 \approx 0.094$
6	1	$1/64 \approx 0.016$

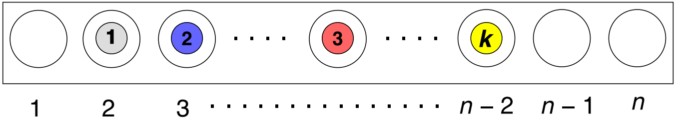
An n-row Plinko



- $k =$ bucket number.
- To reach bucket k, ball must make k turns to the right and $n - k$ turns to the left.

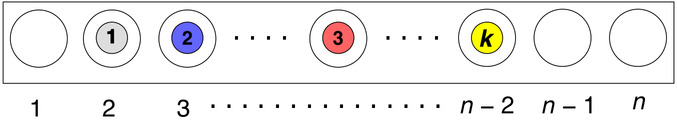
Beans and Cups

- For an n-row plinko, the number of paths to bucket k is the number of ways to place k labeled beans in n cups **in a single order**.



- To calculate this number:
  1. Calculate the number of ways to place k labeled beans in n cups, **in any order**.
  2. Calculate the number of ways to place k labeled beans in k cups, in any order.
  3. Divide result of 1 by result of 2.

The Number of Ways to Place k Labeled Beans in n Cups, in a Single Order



- The number of ways to place k labeled beans in n cups in <u>any</u> order:

$$n(n-1)(n-2)\cdots(n-k+1) =$$

$$\frac{n(n-1)\cdots(n-k+1)(n-k)(n-k-1)\cdots 2\cdot 1}{(n-k)(n-k-1)\cdots 2\cdot 1} = \frac{n!}{(n-k)!}$$

- The number of ways to place k labeled beans in k cups in any order ($k = n$):

$$\frac{k!}{(k-k)!} = k!$$

- The number of ways to place k labeled beans in n cups **in a single order** is:

$$\frac{n!}{(n-k)!}\cdot\frac{1}{k!} = \frac{n!}{k!(n-k)!}$$

Clicker Question #1

For a 5-row plinko, with 6 buckets labeled 0 to 5, how many paths are there to bucket 3?

A) 2

B) 4

C) 6

D) 8

E) 10

$$\frac{5!}{3!(5-3)!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(3 \cdot 2 \cdot 1)(2 \cdot 1)} = \frac{120}{12} = 10$$

n choose k

- The expression we have derived applies to much more than plinkos!
- The expression is often written as:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

  and spoken as "n choose k"
- From n objects, choose k of them (each only once) and either
  - Only a single order is allowed (*e.g.,* turns in the plinko)

    Or
  - The order doesn't matter (*e.g.,* unlabeled beans).

    But, not if
  - The objects are labeled and all orders are allowed.

Binomial Coefficients

- The series of numbers generated by

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

  for a specific value of n and increasing values of $k \leq n$ are called "binomial coefficients."

- The binomial coefficients arise in algebra in the expansion of a sum of two terms:

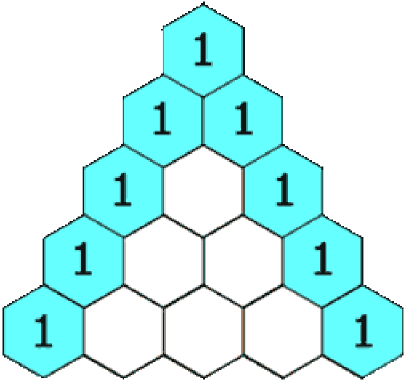
$$
\begin{aligned}
(a+b)^2 &= a^2 + 2ab + b^2 \\
(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\
(a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\
(a+b)^5 &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 \\
(a+b)^6 &= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6
\end{aligned}
$$

Pascal's Triangle

$$
\begin{array}{ccccccccccccc c}
 & & & & & & & & & & & & & \text{Row} \\
 & & & & & & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
 & & & & & 1 & & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
 & & & & 1 & & 2 & & 1 & \cdots & \cdots & \cdots & \cdots & 2 \\
 & & & 1 & & 3 & & 3 & & 1 & \cdots & \cdots & \cdots & 3 \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & \cdots & \cdots & 4 \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \cdots & 5 \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \cdots & 6
\end{array},
$$

- Blaise Pascal, French mathematician, 1623–1662
- Triangle was known long before Pascal's time, but Pascal wrote a book about it.

Pascal's Triangle



- Start with 1s on left and right sides.
- Calculate other elements by adding two values above.

Animation from `https://en.wikipedia.org/wiki/Pascal%27s_triangle`

Probabilities for an n-row Plinko

- The total number of paths is 2^n.
- If each turn to the right or left is equally probable, the probabilities of all paths are equal, and the probability of each path is:

$$p = \frac{1}{2^n} = 2^{-n}$$

- The probability of a ball landing in bucket k is the number of paths to the bucket multiplied by the probability of each path:

$$p(k) = \frac{n!}{k!(n-k)!} \cdot 2^{-n}$$

Clicker Question #2

For a 7-row plinko, with 8 buckets labeled 0 to 7,
what is the probability of a ball landing in bucket 1?

(There's a hard way and an easy way!)

A) ~ 0.01

B) ~ 0.05

C) ~ 0.1

D) ~ 0.15

E) ~ 0.2

$$p(1) = \frac{n!}{k!(n-k)!} \cdot 2^{-n} = \frac{7!}{1!(7-1)!} \cdot 2^{-7} = \frac{7!}{6!} \cdot 2^{-7} = 7 \cdot 2^{-7}$$

What if the Plinko is Biased?

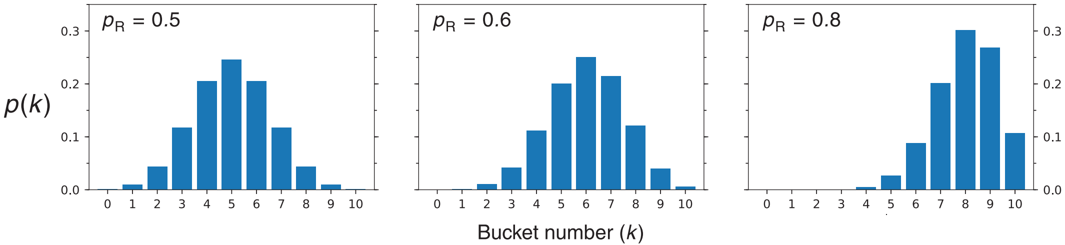
- Suppose that each peg in the plinko has been “fixed”, so that the probability of a left turn is 0.4 and the probability of a right turn is 0.6.
- For each of the paths to bucket k, there are k right turns and $(n - k)$ left turns.
- For each individual path to bucket k, the probability is:

$$0.6^k \times 0.4^{(n-k)}$$

- The total probability of a ball falling in bucket k is:

$$p(k) = \frac{n!}{k!(n-k)!} \times 0.6^k \times 0.4^{(n-k)}$$

Probabilities for a Biased 10-row Plinko



- Biased pegs "push" balls to the right.
- Probability (number of paths) "draws" balls to the center.
- Can you think of physical processes like this?

The Binomial Probability Distribution Function

- The general formulation:
  $p(k; n, p)$ is the probability of k successes in n successive binary (yes/no) trials when the probability of success in each trial is p.
- The probability function:

$$p(k; n, p) = \frac{n!}{k!(n-k)!} p^k (1-p)^{(n-k)}$$

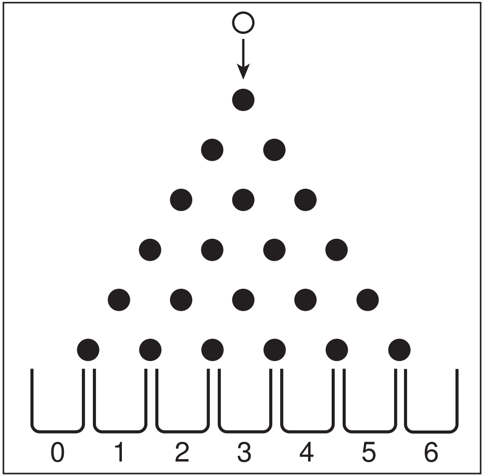
- Some applications beyond plinkos:
  - Number of heads in n successive coin tosses.
  - Number of successes in prescribing a medication to a series of patients with the same condition.
  - Probability of surviving n potentially deadly events.
    $p(n; n, p)$, were p is the probability of surviving each event

Playing Plinko for Cash

- Suppose that I let you put a ball in the 6-row plinko, and I agree to pay you k dollars if the ball lands in bucket k.
- This is probably going to cost me money!
- How much should I charge you to play?
- How much, on average, am I going to have to pay?

Clicker Question #3

How much should I charge you to play my plinko game (to break even)?
All answers count for now.



A) $1

B) $2

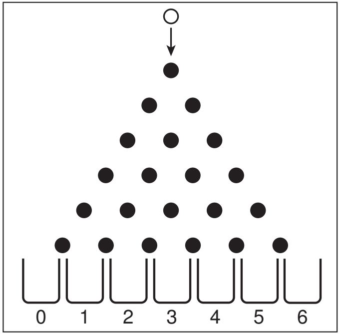
C) $3

D) $4

E) $6

F) $7

An Quick Solution



- Buckets 0 and 6 have equal probabilities. The average payout for these two is $3.
- Buckets 1 and 5 have equal probabilities. The average payout for these two is $3.
- Buckets 2 and 4 have equal probabilities. The average payout for these two is $3.
- The payout for bucket 3 is $3.
- The overall average payout is $3.

- Without knowing any of the actual probabilities!

Random Variables

- Definition: A variable that is assigned a value for each possible outcome or event for a probabilistic process.
- Examples:
  - For a coin toss, we could assign a random variable, x, the value of 1 for heads or 0 for tails.
  - For n successive coin tosses, we could define x to be the number of heads.
  - For the Plinko, we can define the random variable, x, as the number of the bucket that the ball lands in.
    But, we could define other random variables, too.