Physical Principles in Biology
Biology 3550
Fall 2018

Lecture 9:

Plinko Probabilities, Part III

Random Variables, Expected Values and Variances

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The Binomial Probability Distribution Function

- The general formulation:
  $p(k; n, p)$ is the probability of k successes in n sequential binary (yes/no) trials when the probability of success in each trial is p.
- The probability function:

$$p(k; n, p) = \frac{n!}{k!(n-k)!} p^k (1-p)^{(n-k)}$$

- Some applications beyond plinkos:
  - Number of heads in n successive coin tosses.
  - Number of successes in prescribing a medication to a series of patients with the same condition.
  - Probability of surviving n potentially deadly events.
    $p(n; n, p)$, were p is the probability of surviving each event

Playing Plinko for Cash

- Suppose that I let you put a ball in the 6-row plinko, and I agree to pay you k dollars if the ball lands in bucket k.
- This is probably going to cost me money!
- How much should I charge you to play?
- How much, on average, am I going to have to pay?

Random Variables

- Definition: A variable that is assigned a value for each possible outcome or event for a probabilistic process.
- Examples:
  - For a coin toss, we could assign a random variable, x, the value of 1 for heads or 0 for tails.
  - For n successive coin tosses, we could define x to be the number of heads.
  - For the Plinko, we can define the random variable, x, as the number of the bucket that the ball lands in.
    But, we could define other random variables, too.

The Expected Value or Expectation

For a random process that has n possible outcomes
(or a complete set of n non-overlapping events):

- The random variable, x, has values of x_k for $k = 1, 2, 3 \ldots n$
- The n possible outcomes (or events) have probabilities of $p(k)$, for $k = 1, 2, 3 \ldots n$
- The expected value of the random variable, x, is defined as:

$$E(x) = \sum_{k=1}^{n} p(k) x_k$$

- If the process is repeated a large number of times, the average value of x will approach $E(x)$.
- For a game of chance, if x_k is the number of dollars paid out for outcome (or event), k, $E(x)$ is the average payout.

Expected Value of x for the Unbiased Six-row Plinko

Bucket	x	$p(x)$	$p(x)x$
0	0	1/64	0
1	1	6/64	6/64
2	2	15/64	30/64
3	3	20/64	60/64
4	4	15/64	60/64
5	5	6/64	30/64
6	6	1/64	6/64
Total		1	192/64 =3

Clicker Question #1

If the pegs in the six-row plinko are modified so that the probability of a turn to the right is 0.6, what will the expected value of x be?

A) Less than 3

B) 3

C) Greater than 3

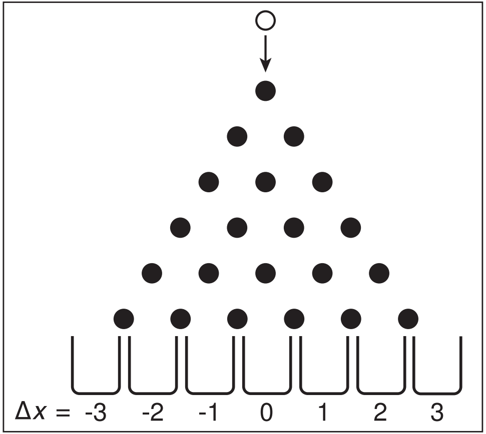
Expected Value of x for a Biased Six-row Plinko: $p(\text{right}) = 0.6$

Bucket	x	$p(x)$	$p(x)x$
0	1	0.004	0
1	1	0.037	0.037
2	2	0.138	0.276
3	3	0.276	0.829
4	4	0.311	1.244
5	5	0.186	0.933
6	6	0.046	0.280
Total		1	3.6

Expected Value of x for a Biased Six-row Plinko: $p(\text{right}) = 0.4$

Bucket	x	$p(x)$	$p(x)x$
0	0	0.046	0
1	1	0.187	0.187
2	2	0.311	0.622
3	3	0.276	0.829
4	4	0.138	0.553
5	5	0.037	0.184
6	6	0.004	0.0246
Total		1	2.4

Another Random Variable for the Plinko, Δx



- Δx represents the position of the bucket, relative to the central bucket.
  $\Delta x = x - 3$

Expected Value of Δx for the Unbiased Six-row Plinko

Bucket	Δx	$p(\Delta x)$	$p(\Delta x)\Delta x$
0	-3	1/64	-3/64
1	-2	6/64	-12/64
2	-1	15/64	-15/64
3	0	20/64	0
4	1	15/64	15/64
5	2	6/64	12/64
6	3	1/64	3/64
Total		1	0

Expected Value of Δx for a Biased Six-row Plinko: $p(\text{right}) = 0.6$

Bucket	Δx	$p(\Delta x)$	$p(\Delta x)\Delta x$
0	-3	0.004	-0.012
1	-2	0.037	-0.074
2	-1	0.138	-0.138
3	0	0.276	0
4	1	0.311	0.311
5	2	0.186	0.373
6	3	0.046	0.139
Total		1	0.6

Notice:

- For the unbiased six-row plinko:

$$E(x) = 3$$
$$E(\Delta x) = 0$$

- For the biased six-row plinko:

$$E(x) = 3.6$$
$$E(\Delta x) = 0.6$$

- For both, $E(\Delta x) = E(x) - 3$

$$\Delta x = x - 3$$

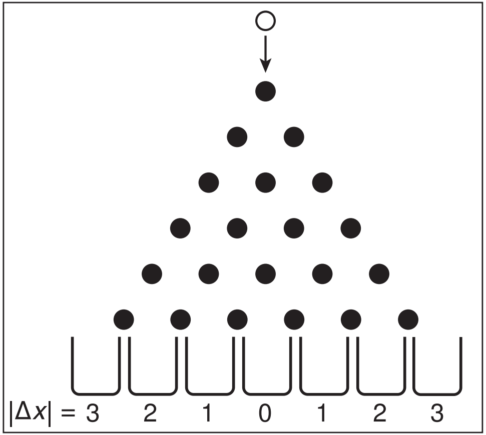
- In general, if x is a random variable, and a is a constant:

$$E(x + a) = E(x) + a$$

- Also:

$$E(ax) = aE(x)$$

Another Random Variable for the Plinko, $|\Delta x|$



- $|\Delta x|$ represents the distance from the central bucket.

Expected Value of $|\Delta x|$ for the Unbiased Six-row Plinko

Bucket	$\|\Delta x\|$	$p(\|\Delta x\|)$	$p(\|\Delta x\|)\,\|\Delta x\|$
0	3	1/64	3/64
1	2	6/64	12/64
2	1	15/64	15/64
3	0	20/64	0
4	1	15/64	15/64
5	2	6/64	12/64
6	3	1/64	3/64
Total		1	$60/64 \approx 0.94$

Expected Value of $|\Delta x|$ for a Biased Six-row Plinko: $p(\text{right}) = 0.6$

Bucket	Δx	$p(\Delta x)$	$p(\Delta x)\Delta x$
0	3	0.004	0.012
1	2	0.037	0.074
2	1	0.138	0.138
3	0	0.276	0
4	1	0.311	0.311
5	2	0.186	0.373
6	3	0.046	0.139
Total		1	1.05

Clicker Question #2

What is the expected value of $|\Delta x|$ for a biased six-row plinko with $p(\text{right}) = 0.4$?

A) -1.05

B) -0.94

C) 0

D) 0.94

E) 1.05

Expected Value of $|\Delta x|$ for a Biased Six-row Plinko: $p(\text{right}) = 0.4$

Bucket	$\|\Delta x\|$	$p(\|\Delta x\|)$	$p(\|\Delta x\|)\,\|\Delta x\|$
0	3	0.046	0.140
1	2	0.187	0.373
2	1	0.311	0.311
3	0	0.276	0
4	1	0.138	0.138
5	2	0.037	0.074
6	3	0.004	0.012
Total		1	1.05

Two Important Parameters for any Discrete Random Variable

- Expected value, also called the mean (μ)

$$\mu = \sum_{k=1}^{n} p(k) x_k = E(x)$$

- Variance (σ^2)

$$\sigma^2 = \sum_{k=1}^{n} p(k)(x_k - \mu)^2$$

  - A measure of the width of the distribution of x values around the mean.
  - Mean of the squares of the differences between x-values and the mean.
  - Squares are taken so that both positive and negative differences contribute.
  - Square root of the variance, σ, is called the standard deviation. σ has the same dimensions as x and μ.

Mean, Variance and Standard Deviation for the Binomial Probability Distribution Function

- The probability function:

$$p(k; n, p) = \frac{n!}{k!(n-k)!} p^k (1-p)^{(n-k)}$$

for $k = 0$ to n.

- The mean of k:

$$\mu = np$$

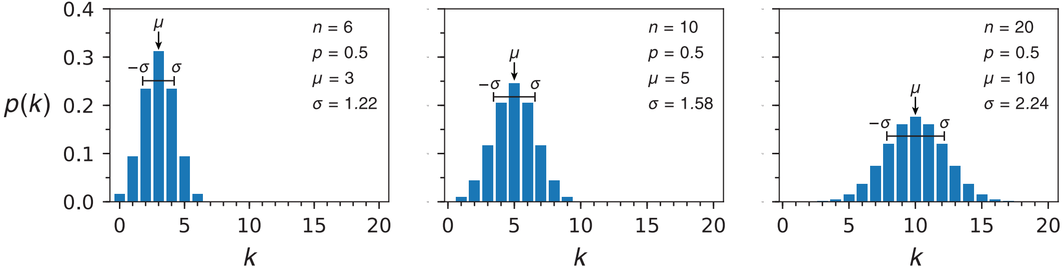
- The variance:

$$\sigma^2 = np(1-p)$$

- The standard deviation:

$$\sigma = \sqrt{np(1-p)}$$

Effect of n on the Mean and Standard Deviation for the Binomial Probability Distribution Function

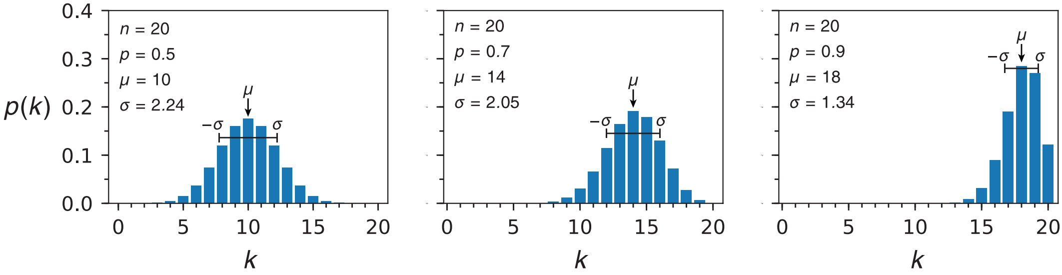


$$\mu = np$$

$$\sigma = \sqrt{np(1-p)}$$

- The distribution gets wider as n gets larger.

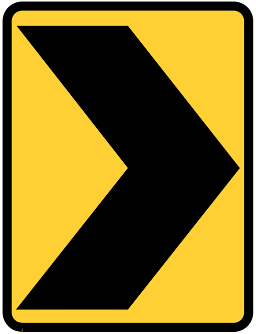
Effect of p on the Mean and Standard Deviation for the Binomial Probability Distribution Function



$$\mu = np$$

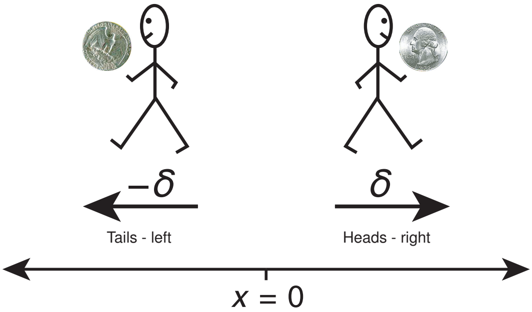
$$\sigma = \sqrt{np(1-p)}$$

Warning!



Direction Change

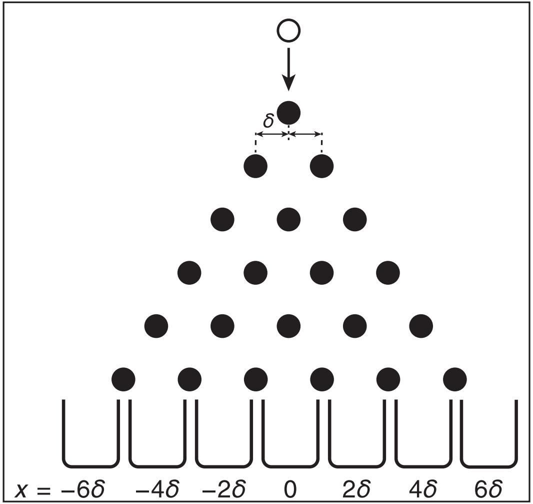
A Random Walk in One Dimension



1. Start at position $x = 0$.
2. Flip coin.
   - Heads, take step of length δ to the right.
   - Tails, take step of length δ to the left.
3. Repeat 2 another $(n - 1)$ times.

- Final position is $x(n)$.
- Generally expect a distribution of $x(n)$ if the random walk is repeated a large number, N, of times.

Like a Plinko, with variable x



- x represents the position of the bucket, relative to the central bucket.