December 30, 2024 Draft

Physical Principles in Biology

Biology 3550/3551

Spring 2025

Chapter 1:
The Scale of Things; Units and Dimensions

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Chapter 1

The Scale of Things: Units and Dimensions

Aside from establishing links among the various sciences, a goal of this class is to help strengthen some of the skills that are required in all of the sciences, especially quantitative skills. Working with dimensions and units is one of the most important of these skills.

Historically, one thing that has tended to distinguish biology from the physical sciences is the extent to which mathematics is used. This distinction is diminishing, but there is certainly a strong tradition in biology that is very descriptive. The greatest of all biologists was (arguably) Charles Darwin, who used little or no mathematics.

What is so good about using mathematics in science? Is there anything that Darwin should have used math for? Two major things that math brings to biology and other sciences are that:

- Math provides a way to formalize a description, or "model" a phenomenon.
- Mathematical models have the power of prediction, both to test the theory and make useful predictions. Predictions are at the heart of the current debate about climate change (at least at one level). How good are the predictions?

Interestingly, Darwin's successors, evolutionary biologists, are among current biologists who use math most extensively. For instance, determining the evolutionary relationships among different species using DNA sequence data is a major application.

Most, but not all, applications of math in science involve measurable quantities, such as length, area, volume, mass, time, or concentration. Thus, working with the units of these measurements is one of the most important basic math skills for scientists, and you have, no doubt, had experience with this in many of your classes. None the less, many students continue to find this kind of calculation challenging, and I want to spend some time on this subject before moving on in the class. Even if you are already comfortable with this kind of calculation, you may find that there are some interesting subtleties that you may not have thought about before.

1.1 Measurements as comparisons

Although most measurements are expressed in terms of units, such as meters, grams, liters, *etc.*, mathematics usually deals just in numbers. How do we bridge measurement and numbers? To start, it is useful to consider that nearly all measurements involve comparisons. For instance, we measure length by comparison with some sort of ruler, as illustrated in Fig. 1.1A. Similarly, we measure mass by comparing the gravitational force on an object (its "weight") with the gravitational force of a reference mass (Fig. 1.1B).

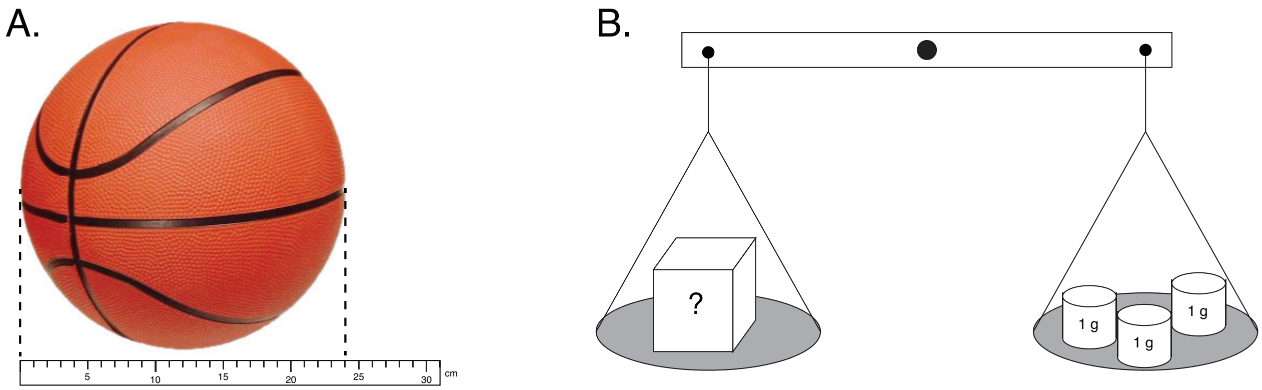

Figure 1.1 Measurement of the diameter of a basketball (A) and the mass of a cube of something (B). In each case the measurement is based on a comparison with some reference object.

The object that we use for comparison then defines the units for the measurement. What is the most natural unit for length? A human body or part of a human body! In the United States, we still use the foot as a unit of length, and the basic unit of length in the metric system, the meter, is on the order of the length of human body.

The somewhat arbitrary nature of unit definitions is illustrated by a famous prank played at the Massachusetts Institute of Technology (MIT) in 1958. A fraternity at MIT decided to use the body of a freshman pledge, Oliver Smoot, to measure the length of the Harvard Bridge, which connects Cambridge, near the site of MIT, to Boston on the other side of the Charles River (and isn't actually close to Harvard University). To do so, they laid Smoot down with his feet at one end of the bridge and his head pointed towards the other. They then made a mark indicating the position of the top of his head, moved his feet to this position and repeated the process until his head reached the other side of the bridge. From this process, the length of the bridge was determined to be 364.4 smoots, ±1 ear[1] , as commemorated in a plaque shown in Fig. 1.2.

The punch line to this story is that Oliver Smoot went on to a distinguished career in the discipline of *metrology*, the science and technology of measurement. He served as both chairman of the American National Standards Institute (ANSI) and president of the International Organization for Standardization (ISO), the U.S. and international agencies that define the standards of measurement.

Although any unit for length, as an example, can be used as a reference for length at any scale, it is convenient to have units that are appropriate for different ranges. In addition, it is very helpful to have your own "internal rulers" to aid in thinking about the very different scales that we encounter in the sciences, especially when we can't directly experience them on the scales of our body. Fig. 1.3 shows a few examples of biological and manufactured objects on a wide range of length scales. In this figure, lengths are indicated both in meters (at the top) and in units derived from the meter.

[1]The names of units are never capitalized, even when they are derived from a person's name, such as the newton or tesla (a unit of magnetic field strength, not the car). On the other hand, the abbreviations of these names are capitalized, such as N, T or (I presume) S.

Figure 1.2
The plaque commemorating the measurement of the the length of the Harvard Bridge in smoot units.

https://en.wikipedia.org/wiki/Smoot

https://alum.mit.edu/news/AlumniNews/Archive/smoots_legacy

Some typical lengths that are relevant in biology are:

- 1 meter (m) $\approx$ length of an adult human arm
- 1 millimeter (mm) $= 10^{-3}$ m $\approx$ length of some of the smallest multicellular animals, *e.g.,* the nematode *C. elegans*. Also about the diameter of a sharp pencil point.
- 1 micrometer (μm or just μ) $= 10^{-6}$ m $\approx$ length of a bacterial cell.
- 1 nanometer (nm) $= 10^{-9}$ m $\approx$ radius of a small protein molecule.
- 1 angstrom (Å) $= 10^{-10}$ m $= 0.1$ nm $\approx$ length of a covalent chemical bond.

All but the last of the units of listed above use the meter and one of the standard prefixes defined in the metric system. (We will get more specific about what we usually mean by the term "metric system" shortly.) The same prefixes are used for nearly all metric units and, you should be or become fluent in using the ones listed in Table 1.1 on the following page.

Although it may seem an arcane subject, the definition of units and the establishment of standards is of immense practical importance for science, technology and commerce. In the United States, Article 1, Section 8 of the Constitution gives Congress the power (among other things) "To coin Money, regulate the Value thereof, and of foreign Coin, and fix the Standard of Weights and Measures." To fulfil this responsibility, in 1830 Congress established the Office of Standard Weights and Measures, as part of the Department of the Treasury. This office was replaced 1901 by the National Bureau of Standards (NBS). Over time, the NBS took on a broader range of activities, and in 1988 it was replaced by the National Institute of Standards and Technology (NIST) and is now part of the Department of Commerce. Other nations have comparable agencies, and they collaborate to establish international standards through the International Organization for Standardization (ISO)[2]

[2] In English speaking countries it is often believed that ISO stands for International Standards Organization and should be pronounced "eye-ess-oh". But, ISO is defined by the organization as an official abbreviation to be used in all languages and pronounced "iso", which is derived from the Greek word for equal, isos. (I usually forget and say "eye-ess-oh", but it's wrong!)
https://en.wikipedia.org/wiki/International_Organization_for_Standardization

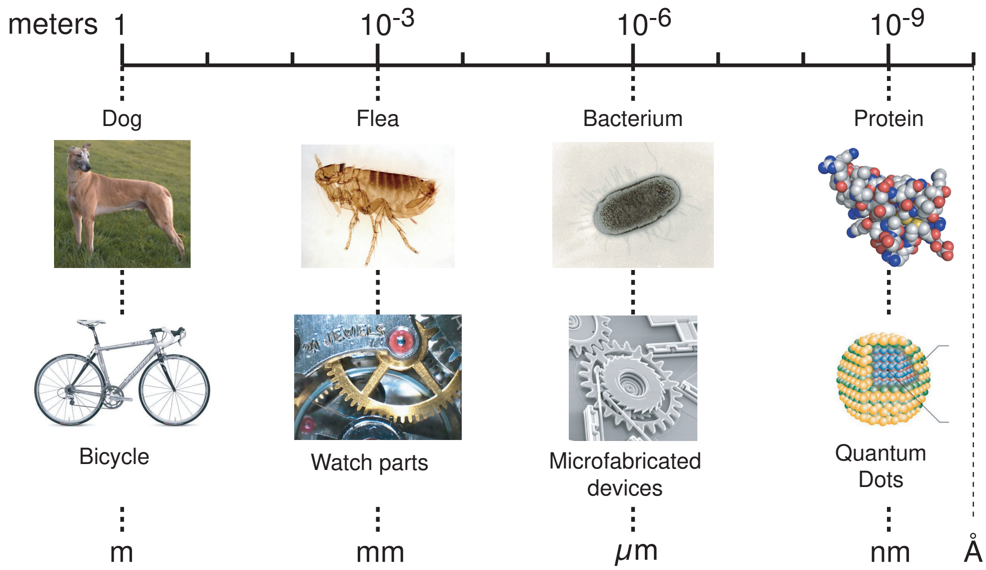

Figure 1.3 Some examples of biological and fabricated objects on a wide range os size scales.

Table 1.1 Standard prefixes for units in the SI metric system. See http://physics.nist.gov/cuu/Units/prefixes.html for prefixes covering the range from 10^{-24} to 10^{24}.

prefix	abbreviation	multiplier	examples
nano	n	10^{-9}	nm, ng
micro	μ	10^{-6}	μm, μg
milli	m	10^{-3}	mm, mg
centi	c	10^{-2}	cm, cg
deci	d	10^{-1}	dm, dg
kilo	k	10^{3}	km, kg
mega	M	10^{6}	Mm, Mg

1.2 Units versus dimensions and a brief history of the metric system

The types of quantities described in the previous sections imply a built in reference object for comparison, such as an object 1 m long. These quantities are called *units* and they are distinguished from another kind of quantity called *dimensions*. A dimension is a quantity like length, mass *etc.* that can be expressed in different, but interchangeable, units. We can compare 1 m and 1 smoot, but we can't compare 1 m and 1 g. (In principle we could have a unit of mass defined as the mass of Oliver Smoot, but this would be very confusing!) Quantities that can be compared directly, such as length, have the same dimension, even if they are given in different units, such as 1 km and 1 mile.

Modern systems of measurement recognize (a minimum of) 5 basic dimensions:

- Length, L
- Mass, M
- Time, T
- Temperature, Θ
- Electric charge, Q, or current, I

Note that the symbols for these quantities are written with italic (or Greek) characters, which follows the mathematical typesetting convention that most variables are represented in this way. The dimensions are usually used in more abstract expressions where specific values are not assigned.

Other dimensions can be derived from the five listed above. Some examples are listed below.

- Area: $A = L^2$
- Volume: $V = L^3$
- Velocity: distance per unit time: $v = L/T$
- Acceleration: change in velocity per unit time: $a = (L/T)/T = L/T^2$
- Force: defined by Newton's second law of motion, $f = m \cdot a$: $f = ML/T^2$

Notice that the dimension of volume is defined in terms of length, as L^3, but the liter (L) is a *unit* of volume. This is a particularly tricky case where it is important to make sure we are talking about a unit or a dimension. Also, the liter is somewhat of an oddity, because its abbreviation is an upper-case letter but is not derived from a person's name.

For most of history, measurements have been made using a mish-mash of units chosen for different purposes, such as cubits, furlongs, feet, miles, *etc.*. This is still true to a degree, but far less so.

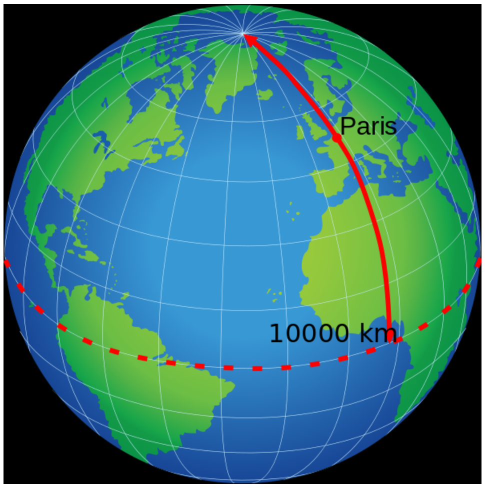

Figure 1.4
The original definition of the meter (1795) as as one ten-millionth (10^{-7}) of the distance from the Equator to the North Pole, along the meridian passing through Paris. Illustration from: https://en.wikipedia.org/wiki/Metric_system

1.2.1 Early metric systems

Although there were earlier precedents, the origins of our current metric system lie in the French Revolution of 1789–1799. Although you might not think that the details of measurement would be an important issue in a political revolution, one of the grievances that led to the revolution was inconsistency among tax collections in different parts of France and with other countries, in part because of the use of numerous different units for measurement of goods. The leaders of the French Revolution also placed a high value on rationality (as they saw it) and wanted a measurement system based on powers of 10. They even went so far as to introduce a decimal calendar and clock. Although the decimal time system didn't catch on, the decimal metric system definitely did, and the United States is now one of just a few countries that use derivatives of what are commonly called "English" units (which, themselves, are not entirely consistent among the countries that use them).

In the 1790s, the French defined two basic (as we would consider them now) units, the *métre* (meter in English) and the *gramme* (gram). Traditionally the definition of any unit has required some standard object that can be used for comparison, and the best standard objects are the most universal ones, so that they are accessible, in principle, to anyone. In this vein, the French chose the Earth itself as the reference object for length, and they defined the meter as one ten-millionth (10^{-7}) of the distance from the Equator to the North Pole, along the meridian passing through Paris, as illustrated in Fig. 1.4. The gram, in turn, was defined as the mass of water in a volume defined by the meter, specifically $1\,\mathrm{cm}^3 = 10^{-6}\,\mathrm{m}^3$.

Although these definitions are clear and based on well-defined physical objects, the definition of the meter, especially, is obviously problematic for practical measurements of, say, the height of a house. Not only is the specified distance along the surface of the Earth absurdly impractical for comparison to a house, the distance was not very well-determined at the time. To address the latter problem, a survey expedition was commissioned to measure a fraction of a meridian that *almost* passes through Paris (from Dunkirk, France to Barcelona, Spain). This turned out to be a rather heroic endeaver that lasted seven years, and the calculations ignored what are now recognized to be significant asymmetries in the shape of the Earth. None the less, the survey produced a defined distance. To create a practical reference for the meter, a platinum bar, the *mëtre des Archives*, based on the meridian measurement was fabricated and placed for safekeeping in the French National Archives. This reference was then used to create secondary references, which were then used to make additional references, and so on. As it turned out, the platinum bar was about 0.02 m shorter than defined by the meridian, but it remained the standard reference for many decades, probably because everyone was so tired of the whole business by then!

At the same time, a reference object for the gram, a cylinder of platinum with mass equal to that of $1000\,\mathrm{cm}^3$ of water, representing 1000 g=1 kg, was fabricated and placed in the French National Archives. This object was designated the *kilogramme des Archives*. The French also defined units for area and volume based on the meter.

In 1875, an international treaty, the *Convention du Mëtre* was signed by 17 nations and called for replacement of the *mëtre des Archive* and the *kilogramme des Archive* with new standard reference objects, the international prototype meter (IPM) and the international prototype kilogram (IPK). Like the earlier reference objects, these were made of a platinum alloy and were based on the French standards. Importantly, however, the meter and the kilogram were now defined directly in terms of the IPM and IPK, as opposed to the length of a meridian or the mass of a given volume of water (or other substance). This eliminated any question of how closely the reference objects matched the the official definitions of the units. On the other hand, this placed extraordinary importance on the objects themselves, and the *Convention du Mëtre* and the organizations it created, established detailed protocols for maintaining the standards and replicating them. Copies of the IPM and IPK were made and distributed to all of the treaty signatories, which then used these as references for their own countries. The *Convention du Mëtre* also established an international organization to continue work on refining measurement standards and to organize conferences for this purpose, the General Conference on Weights and Measures (Conférence générale des poids et mesures, CGPM). Since then, there have been 26 CGPM meetings.

The IPM remained the definition of the meter until 1960, when the meter was redefined in terms of the wavelength of light corresponding to a specific electronic transition in krypton 85 atoms. This represented the long-sought goal of a standard that was independent of a single object and was, in principle, accessible anywhere. However, the IPK continued to serve as the international standard for mass until 2018, when a major revision of the SI was adopted, as discussed further below.

During the century following the establishment of the French metric system (and related ones in other countries), some of the major advancements in the physical sciences were in the fields of thermodynamics, electricity and (closely related to electricity) magnetism. These fields required the the development of entirely new classes of dimensions and units. It took much of the nineteenth century to define the basic quantities for these disciplines, and even then there were two basic approaches for relating electricity and magnetism to force as defined by Newton's second law[3]. In 1873, a committee of the British Association for the Advancement of Science proposed a system of units that unified units for electricity and magnetism with previously defined units for length, mass, time and temperature. This system came to be known as the centimeter-gram-second (cgs) system, which was widely

[3]The two approaches differed by whether force was related to electricity in terms of the electrostatic interaction between two charges or the magnetic interaction between currents flowing through two wires.

adopted and was the official "metric system" for several decades. Even with the introduction of the cgs system, however, there were internal inconsistencies involving electrostatic and magnetic forces. As a consequence, there were actually two branches of the cgs system (each with further variants). This can still be a source of confusion, especially when reading older publications and trying to convert values to the modern conventions.

1.2.2 Establishment of the Modern Metric System, the Système International d'unitès (SI) and Further Revisions

The next major revision to the metric system was enacted in 1960, when the name Système International d'unitès (SI) was introduced at the 11$^{\text{th}}$ CGPM.. Among other things, the SI finally settled on a single, consistent way of dealing with electricity and magnetism. The SI also replaced the centimeter and gram as the basic units of length and mass, respectively, with the meter and kilogram. As a consequence, the SI is sometimes referred to as the MKS (meter, kilogram, second) system, but this is really a more general designation that includes some predecessors to the SI.

Rapid developments in physics and electronics (especially the invention of the laser) led to another redefinition of the meter in 1983. The meter is now defined as the distance travelled by light in a vacuum during 1/299792458 of a second.

This redefinition of the meter was more profound than earlier changes in units, because it depended on establishing an essentially arbitrary definition of a fundamental constant of the universe, the speed of light, as opposed to a measurement of the constant using defined units. At the same time that the meter was redefined, the speed of light was declared to be exactly 299792458 m/s. The second had previously been defined, in 1967, as the time of 9,192,631,770 cycles of the radiation associated with a specific quantum transition[4] of cesium 133 (^{133}Cs) atoms (The frequency of this radiation is designated $\Delta\nu_{\mathrm{Cs}}$). Like the speed of light, the second was defined by setting the value of a physical constant, in this case $\Delta\nu_{\mathrm{Cs}}$. Although the numbers associated with these definitions may seem rather arbitrary and not very convenient, they were chosen to give the best possible match to the original standards.

The new definition of the meter represented a change from an *explicit-unit* to an *explicit-constant* basis for defining units. Though this indirect approach is admittedly rather awkward, it allows the continuous refinement of numerical values of units by making more precise physical measurements of the constants. For instance, if the speed of light were to be more precisely measured, the official value for this constant will remain exactly 299792458 m/s, but the standard for the meter or second would be adjusted to reflect the improved measurement. Furthermore, these measurements can, in principle, be made by anyone in any location. For instance, it would be possible to for an extraterrestrial civilization to implement our definition of a meter, provided only that they know our definitions and can measure the speed of light and the frequency of the ^{133}Cs transition used to define the second.

In contrast to the definition of the meter, the IPK proved to be remarkably difficult to replace as the standard for mass, which was finally accomplished in 2018. In 2007 the CGPM called for a complete shift in the definition units to an explicit-constant basis. The 2011 CGPM further decided to base the definition of the kilogram on a fixed value of the Planck

[4]The ground-sate hyperfine transition

constant, h, which defines the relationship between the energy of a quantum transition, E, and the frequency, ν, of the electromagnetic radiation absorbed or released during that transition

$$E = h\nu$$

As discussed below, the units of energy are expressed in SI units as $\mathrm{kg \cdot m/s^2}$, so that the units of the Planck constant are given by:

$$h = \frac{E}{\nu} = \frac{\mathrm{kg \cdot m/s^2}}{\mathrm{s^{-1}}} = \mathrm{kg \cdot m/s}$$

Once the Planck constant is given a fixed value (along with the defined values for the meter and second), the kilogram can be defined as:

$$\mathrm{kg} = \frac{h \cdot \mathrm{s}}{\mathrm{m}}$$

The 2011 CPMG defined the value of the Planck constant to be exactly $6.62606 \times 10^{-34}\,\mathrm{kg \cdot m^2 s^{-1}}$. However, the kilogram was not redefined at this time, because there was not yet a method deemed sufficiently precise for implementing the new definition, which requires being able to actually make measurements of mass that are directly related to the Planck constant. The first step in that process was to establish very precise measurements of the Planck constant in terms of the existing definition of the kilogram. The criteria established in 2011 specified that the Planck constant be determined by three independent experiments, using two independent methods, with uncertainties less than 50 parts per billion (5×10^{-8}). At that time, there were two methods available for measuring the Planck constant with such precision, both of which are somewhat round-about.

The first method is based on the value of Avogadro's number (also called the Avogadro constant, N_{A}), the number of atoms, molecules or ions in a mole of a substance. Avogadro's number is directly related to the Planck constant, through other physical constants that have been measured with extremely high precision (less than one part per billion). Thus, a measurement of Avogadro's number is equivalent to a measurement of the Planck constant. Prior to 2019, Avogadro's number was defined as the number of atoms in 12 g of pure carbon 12 (^{12}C). But, determining an actual value for N_{A} directly from that definition is highly problematic, because carbon readily undergoes a variety of chemical reactions and even pure carbon can exist in multiple forms. Instead silicon, which forms very stable and well defined crystals were used.

The method used to determine Avogadro's number is referred to as an x-ray crystal density (XRCD) measurement and used crystals of silicon highly enriched (99.995%) in ^{28}Si. Implementation of the XRCD method involved machining two spheres of crystalline silicone ^{28}Si (to provide two of the three independent measurements called for). The sphere is the three-dimensional shape that can be most precisely manufactured (using a lathe), and its size is defined by a single measurable length, the diameter. The spacing between atoms in the crystal can be determined with very high precision by x-ray crystallography, so that the number of atoms in the sphere can be determined with very low uncertainty. However, to correlate the number of atoms with the mass of the crystal, it was also necessary to determine

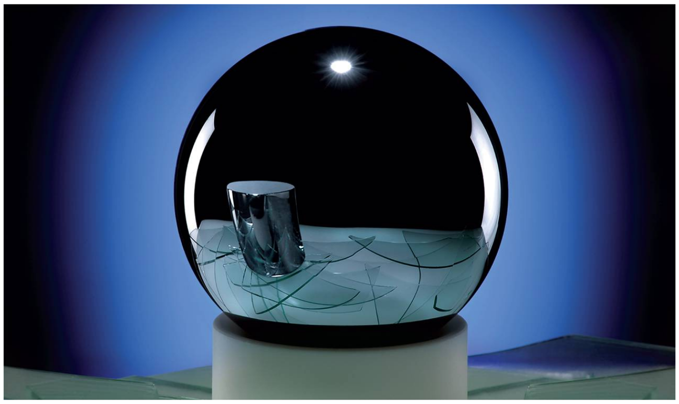

Figure 1.5
One of two silicon spheres, 99.995% enriched in ^{28}Si, used to establish Avogadro's number. A reflection of an old kilogram standard can be seen in the surface of the sphere. Illustration from: Cho, A. (2018). World poised to adopt new metric units. *Science*, 362, 625–626. http://doi.org/10.1126/science.362.6415.625 Photograph from the Physikalisch-Technische Bundesansttalt (PTB), Germany.

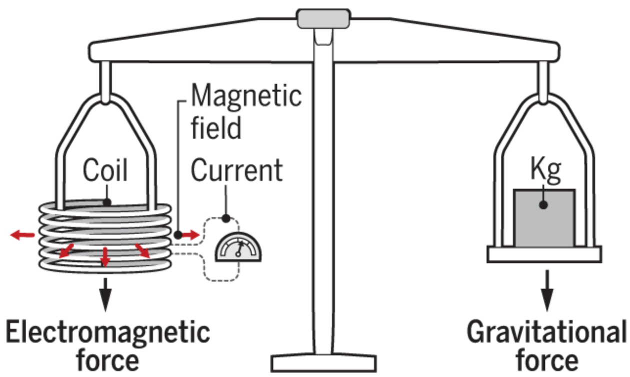

Figure 1.6
Schematic diagram of a Kibble (or moving-coil watt) balance. The gravitational force acting on the mass to be determined (on the left) is balanced by the magnetic force between a stationary magnetic field and the field generated by electric current passing through the coil attached to the right-hand side of the balance. The electric current required to balance the two forces is measured and used to calculate the mass of the object. Adapted from:
Cho, A. (2017). Plot to redefine the kilogram nears climax. *Science*, 356, 670–671. http://dx.doi.org/10.1126/science.356.6339.670

precisely the isotopic composition of the silicon, which was done by mass spectrometry. Over a period of several years, measurements of the dimensions and isotopic composition of these spheres were refined, until the reproducibility could be established to be 1 part in 50 million. Among other precautions, this required extensive polishing to remove virtually all traces of contamination from the surface. These spheres have been described as "the world's roundest objects", and a photograph of one of them is shown in Fig. 1.5. From the measurements of the silicon spheres, the value of Avogadro's number was defined, at the 2018 CGPM, to be exactly $6.02214076 \times 10^{23}$.

The second method for measuring the Planck constant uses a special electronic balance called a *Kibble balance*, illustrated in Fig. 1.6. In this device, the force of gravity acting on the object to be weighed is balanced by the magnetic force generated by a coil of wire in a magnetic field, and the mass is determined by the electric power, expressed in watts, required to balance the two forces. Although this basic idea for a balance is not new (and is commonly incorporated in electronic laboratory balances), an important refinement was introduced in 1975 by Bryan Kibble, who devised a method to internally calibrate the coil and magnetic field. Kibble died in 2016, just before his invention was expected to enable a new definition of the kilogram, and the device (previously described as a moving-coil watt balance) was named in his honor.

The power required to balance the mass represents the product of the electrical current, measured in amperes (A) passing through the coil and the electrical potential across the coil, measured in volts (V). The current and potential can be directly related to the Planck constant through measurements of two quantum-mechanical phenomena, the quantum Hall effect and the Josephson effect. Using a kilogram standard (as then defined), it was possible to precisely measure the Planck constant in a way that is independent of the XRCD analysis of the ^{28}Si spheres.

By 2018, the criteria established in 2011 for measuring the Planck constant, with an uncertainty of less than 50 parts per billion, had been met by multiple experiments using both XRCD method for determining Avogadro's number and the Kibble balance. This milestone, then enabled the 2018 CGPM to finally redefine the kilogram in terms of the Planck constant, the second (defined by $\Delta\nu_{\mathrm{Cs}}$) and the meter (defined by the second and the speed of light). At the same time, Avogadro's number was redefined to be exactly 6.02214076×10^{23}.

In principle, internally calibrated Kibble balances can now be used to independently measure masses in laboratories, or more practically, Kibble balances can be used to measure reference masses that can be used as secondary standards. Alternatively, spheres of ^{28}Si characterized by XRCD could be used a mass references. At present, however, both of these techniques are very challenging in practice, and only a few nations have the resources to implement them. The XRCD method is particularly difficult to implement, and for the near future the Kibble balance is expected to be used to calibrate secondary standards, including the IPK, which will be used to calibrate additional reference masses. As a consequence, the redefinition of the kilogram will have practical consequences at only the highest level of metrological standardization. It should also be noted that the Kibble balance is actually a bit less precise (about $20\,\mu$g/kg) than the best conventional balances (with precisions as low as (1–2 μg/kg), so that reference objects calibrated with the new method are expected to be slightly more variable than before. To place this in perspective, however, the uncertainties from the Kibble balance are on the order of one millionth of one percent.

1.2.3 The base dimensions of the SI and their current definitions

The SI defines seven "basic" dimensions and their standard units, as summarized in Table 1.2. Of the seven basic units, the first five in the table are independent of one another. However, the amount of a substance (mole) and luminous intensity (*candela*) can be defined in terms of the the other five basic dimensions and are not strictly necessary for a complete set of units. They are included in the SI largely as a matter of convenience and consistency with older definitions.

The *luminous intensity* and its SI unit, the candela, may be the dimension and unit that are least familiar. As suggested by its name, luminous intensity is a measure of the brightness of visible light sources and, specifically the intensity along a specific direction. The origin of the candela, and its name, is reflected by the fact that the luminous intensity of a conventional wax candle is approximately 1 candela. (The Latin for *candle* is *candela.*) Until 1948, various countries had different defined units of luminous intensity defined in terms of very specific light sources, including in some cases a wax candle of defined size and composition. The dimension of luminous intensity (and the several other quantities related

Table 1.2 The seven basic dimensions and units defined in the SI. For details, see http://physics.nist.gov/cuu/Units/units.html

Dimension	SI base Unit	Abbreviation
length	meter	m
mass	kilogram	kg
time	second	s
electric current	ampere	A
thermodynamics temperature	kelvin	K
amount of substance	mole	mol
luminous intensity	cendela	c

to light intensity) are also unusual because they were originally used to express a human response to a stimulus, rather than a specific amount of light, as measured for instance by the number of photons of a specified energy.

By 2019, all of the SI basic units were defined by the values of seven physical constants, which are now set to exact values, as listed in Table 1.3. There is not a simple one-to-one relationship between the basic SI units and the constants listed in Table 1.3, as several of the units are defined in a somewhat hierarchical way, as listed in Table 1.4.

From the seven basic units, there are an almost limitless number of derived units that can be used to specify any measured quantity that has so far been conceived. Some examples of derived SI units are listed in Table 1.5.

Although the the basic units in the SI now well established as the foundation for measurement throughout the world, the system continues to be updated to incorporate new technologies and new applications. Going forward, this will involve making decisions about when to update the numerical values used to define the basic units, while keeping the the seven physical constants fixed to their standardized values.

Table 1.3 The seven physical constants used to define the SI basic units, and the their exact, defined values. For details, see http://physics.nist.gov/cuu/Units/units.html

Constant	Symbol	Exact value
The ground-state hyperfine transition frequency of ^{133}Cs	$\Delta\nu_{\mathrm{Cs}}$	$9.192631770 \times 10^{9}$ Hz
The speed of light in vacuum	c	2.99792458×10^{8} m/s
The Planck constant	h	$6.62607015 \times 10^{-34}$ J · s
The elementary charge	e	$1.602176634 \times 10^{-19}$ coulomb
The Boltzmann constant	k	1.380649×10^{-23} J/K
The Avogadro constant	N_{A}	$6.02214076 \times 10^{23}$ mol^{-1}
The luminous efficacy of monochromatic radiation of frequency 540×10^{12} Hz	K_{cd}	683 lm/W

Table 1.4 Definitions of the basic SI units in terms of the seven physical constants listed in Table 1.3. For details, see http://physics.nist.gov/cuu/Units/units.html

Unit	Defined by:
second (s)	Setting $\Delta\nu_{\mathrm{Cs}}$ to be 9.192631770×10^9 in units of s^{-1}.
meter (m)	Setting the speed of light (c) to be 2.99792458×10^8 in units of m/s, with the second defined in terms of $\Delta\nu_{\mathrm{Cs}}$.
kilogram (kg)	Setting the value of the Planck constant to be 6.62607015×10^{-34} in units of $\mathrm{kg}\cdot\mathrm{m/s}$, with the meter and second defined in terms of $\Delta\nu_{\mathrm{Cs}}$ and c.
ampere (A)	The relationship $1\,\mathrm{A} = 1\,\mathrm{coulomb/s}$, with the coulomb defined so that the elementary charge (e) has the exact value $1.602176634\times10^{-19}$ coulomb, and the second is defined in terms of $\Delta\nu_{\mathrm{Cs}}$.
kelvin (K)	Setting Boltzmann's constant (k) to be 1.380649×10^{-23} when expressed in units of $\mathrm{kg}\cdot\mathrm{m}^2\mathrm{s}^{-2}\mathrm{K}^{-1}$, with the kilogram, meter and second defined in terms of $\Delta\nu_{\mathrm{Cs}}$, c and h.
mole (mol)	Making 1 mol equal to Avogadro's number of elementary entities.
candela (cd)	Setting the value of K_{cd} to be 683 when expressed in units of lm/W, which is equivalent to units of $\mathrm{cd}\cdot\mathrm{m}^{-2}\mathrm{kg}^{-1}\mathrm{s}^{2}$.

Table 1.5 Examples of derived SI units.

Dimension	SI unit
Area	m^2
Volume	m^3
Acceleration	$\mathrm{m/s}^2$
Force	newton (N) $= \mathrm{kg}\cdot\mathrm{m/s}^2$
Energy	joule (J) $= \mathrm{N}\cdot\mathrm{m} = \mathrm{kg}\cdot\mathrm{m}^2/\mathrm{s}^2$
Electric charge	coulomb (C) $= \mathrm{A}\cdot\mathrm{s}$

1.2.4 Other Units

The non-metric units still used in the United States for consumer products and some other purposes are sometimes referred to informally as "English units", but this term, like "metric system", actually covers several historic and closely related systems. The units used in the United States are more properly designated the "United States customary units". The other major system of English units is the Imperial system, which was officially established in 1824, and still has limited use in the United Kingdom, Canada and a few other British Commonwealth nations. The United States customary and Imperial systems are largely similar, but with some distinctions.

In order to simplify conversions with the SI, while not deviating too noticeably from the traditional definition of length units, the U.S. Customary yard is defined directly in terms of the meter, as exactly 0.9144 m, which makes the inch exactly 0.0254 m = 25.4 mm. Similarly, the U.S. customary pound is defined as exactly 453.59237 g,[5] and the US gallon (for liquid measure) is defined as exactly 3.785411784 L. There are also more informally defined "English" units of fluid and dry volume, as typically found in food recipes. These are not so formally defined, and probably don't need to be.

1.3 Using units in calculations

For students and practitioners of science, the important point about units and dimensions is that their proper use is a critical skill! Although students in this course should have had lots of experience in this already, I often find that many are rather rusty.

The simplest problems involving dimensions often have the form, "How many feet are there in a kilometer?", and one can find tables that provide instructions such as, "To covert kilometers to miles, multiply by 0.621371." Instructions like this one are often called *conversion factors* and there are many published tables and websites with conversion factors for different units. Most of these are probably correct, though care is sometimes required, especially when the same word is used for different measurements; "ounces" is a particularly confusing one. One convient and quite comprehensive website for conversions is:

http://www.digitaldutch.com/unitconverter/

But, I can provide no guarantee for the reliability of the information on this site! One can also use Google and other web-search engines to quickly look up conversions, with queries such as:

Miles to feet

More authoritative references for conversion factors are provided at the end of this chapter.

The use of units in calculations has a fancy name, *Dimensional Analysis*. The basic idea of dimensional analysis is to treat units as part of the algebraic terms representing quantities.

[5]More specifically, this defines the pound avoirdupois (derived from a French phrase for "weights and measures") in the U.S. Customary system. There are actually two groups of units for mass in this system, the other being the Troy system, which is still sometimes used for precious metals, and even more are found in other English systems.

This idea can be developed quite formally, but for practical purposes this is not necessary. We can think of conversion factors as recipes, such as “To convert kg to g, multiply by 1,000.”. But we can also write them as equations, such as

$$1\,\mathrm{kg} = 1000\,\mathrm{g}$$

We can re-write this equation as

$$\frac{1\,\mathrm{kg}}{1000\,\mathrm{g}} = 1$$

or:

$$\frac{1000\,\mathrm{g}}{1\,\mathrm{kg}} = 1$$

Remember that any number multiplied by 1 (or divided by 1) is itself. This is expressed more formally by the statement, “Multiplication by 1 is an identity operator.” So, if we want to convert 37 kg to g, we can write this algebraically as:

$$37\,\cancel{\mathrm{kg}} \times \frac{1000\,\mathrm{g}}{1\,\cancel{\mathrm{kg}}} = 37000\,\mathrm{g}$$

This isn't very exciting, but the important point is that the units are treated just as any other algebraic term, like a variable x or a, would be. We know that we did things properly because the kg units cancel out properly to give us an answer in g. If we had divided instead of multiplied we would get:

$$37\,\mathrm{kg} \times \frac{1\,\mathrm{kg}}{1000\,\mathrm{g}} = 0.037\,\mathrm{kg}^2/\mathrm{g}$$

Mathematically, this is correct, but the answer we get doesn't make sense in terms of the units we are looking for.

As a potentially more interesting example, consider the scale and dimension of a bacterial cell. One of the major bacterial species in our gut is *Escherichia coli*, and the cells of this species can be approximately described as cylinders $2\,\mu$m long and $1\,\mu$m in diameter, as illustrated in Fig. 1.7.

Expressing the relationship in terms of dimensions (as opposed to specific units), the volume of a cylinder is calculated as:

$$V = L \times A$$

where L is the length of the cylinder and A is the area of the “caps” at each end. From the equation for area of a circle,

$$V = L\pi R^2$$

where R is the radius of the cylinder. For our bacterium, L=$2\,\mu$m and, R=$0.5\,\mu$m, so that we can replace the dimensions with values with specific units:

$$\begin{aligned} V &= 2\,\mu\mathrm{m} \cdot \pi \cdot (0.5\,\mu\mathrm{m})^2 \\ &= \pi \cdot 0.5\,\mu\mathrm{m}^3 \\ &\approx 1.57\,\mu\mathrm{m}^3 \end{aligned}$$

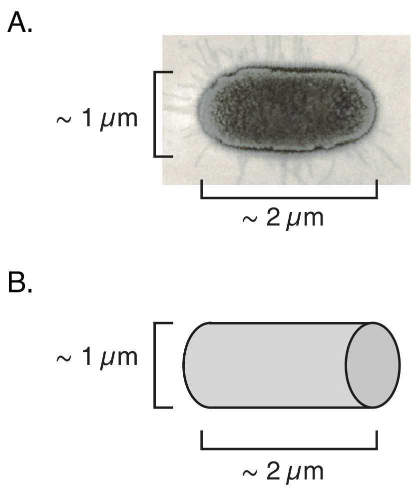

Figure 1.7
A. Light-microscope image of an E. coli bacterium, from http://eduspace.free.fr/ibbiology2007_14/02_cells/e_coli.html.
B. Approximation of the bacterial cell as a cylinder.

So the answer is $\approx 1.6\,\mu\text{m}^3$, but cubic micrometers are not units of volume that are very easy to relate to!

A more conventional unit of volume is a liter. So, how do we get from μm^3 to L? An easy to remember conversion factor for volume is based on the cubic centimeter, or "cc", which is equal to 1 mL. We can use this to derive a conversion factor from mL to m^3.

$$
\begin{aligned}
&1\,\text{cm} = 0.01\,\text{m} \\
&(1\,\text{cm})^3 = (0.01\,\text{m})^3 \\
&1\,\text{cm}^3 = 10^{-6}\,\text{m}^3 \\
&1\,\text{mL} = 10^{-6}\,\text{m}^3
\end{aligned}
$$

Notice that we start with a relationship between two units of linear distance and raise both sides of the equation to the third power to obtain a relationship between units of volume. Importantly, the entire expression on each side of the second line is raised to the third power, not just the units. This follows the standard rules of algebra, and raising only the units to the third power will lead to an incorrect result. We can then manipulate this result further to obtain the relationship between 1 L and m^3.

$$
\begin{aligned}
&1\,\text{mL} \times 1000 = 10^{-6}\,\text{m}^3 \times 1000 \\
&1000\,\text{mL} = 10^{-3}\,\text{m}^3 \\
&1000\,\cancel{\text{mL}} \times \frac{1\,\text{L}}{1000\,\cancel{\text{mL}}} = 10^{-3}\,\text{m}^3 \\
&1\,\text{L} = 10^{-3}\,\text{m}^3
\end{aligned}
$$

This conversion factor is one that is worth committing to memory, as we will frequently want to relate volumes to lengths on a range of scales.

For the case of the bacterial cell, we can use this factor to express the volume in liters:

$$1.6\,\mu\text{m}^3 \times \left(\frac{10^{-6}\,\text{m}}{1\,\mu\text{m}}\right)^3 = 1.6 \times 10^{-18}\,\text{m}^3$$

$$1.6 \times 10^{-18}\,\text{m}^3 \times \frac{1\,\text{L}}{10^{-3}\,\text{m}^3} = 1.6 \times 10^{-15}\,\text{L}$$

A typical laboratory culture of *E. coli* contains about 10^9 (1 billion) bacteria per mL. The total volume of these bacteria is:

$$10^9\,\text{bacteria} \times 1.6 \times 10^{-15}\,\text{L/bacterium} = 1.6 \times 10^{-6}\,\text{L}$$
$$1.6 \times 10^{-6}\,\text{L} \times 10^3\,\text{mL/L} = 1.6 \times 10^{-3}\,\text{mL}$$

So, about 0.2% of the culture volume is occupied by bacteria.

1.4 Units of Concentration

The concept of concentration is central to chemistry and is critical to much of biology and physics. The particular issues that arise in dealing with units of concentration are thus deserving of some review here. There are a variety of different ways that concentrations can be expressed, but we will focus on the ones that are most common and appropriate for the topics we will cover in this course.

1.4.1 Different ways of expressing concentration

For most purposes, the most convenient units of concentration are those that express the amount of solute present in a given total volume of solution. This amount of solute might be expressed as a mass or as the number of moles, to give units, for instance, of g/L or mol/L (M). In practical terms, these definitions mean that we would make, for instance, a 10 g/L solution by measuring 10 g of the solute and dissolve it in somewhat less than 1 L of solvent and then add more solvent to make up a final volume of 1 L. At first glance, this seems like a rather awkward definition and procedure: It would be easier to make a solution by dissolving 10 g of solute in 1 L of solvent. However, for calculations, it is much easier to work with concentrations defined in terms of the total volume, rather than the amount of solvent used. This is because it is straight forward to calculate the amount of solute in a given volume of solution, or conversely the volume that would contain a given amount of solute. For instance, if we have 50 mL of a 50 g/L solution of glucose in water, we can readily calculate the number of grams in the solution as follows:

$$50\,\cancel{\text{mL}} \times \frac{1\,\text{L}}{1000\,\cancel{\text{mL}}} = 0.05\,\text{L}$$

$$50\,\text{g}/\cancel{\text{L}} \times 0.05\,\cancel{\text{L}} = 2.5\,\text{g}$$

Consider, on the other hand calculating the number of grams in a solution made up by dissolving 50 g of glucose in 1 L of water. We would expect that the total volume of this solution would be greater than 1 L, but knowing how much more requires additional information, and this is not a simple calculation! When compounds, even as liquids, are mixed together in a solution their volumes do not necessarily add together in a simple way. The balance of interactions among the different kinds of molecules can bring some pairs closer together and lead to repulsions between others. As a consequence, the final volume can be either smaller or greater than the sum of two volumes of different liquids. When a solid is dissolved in a liquid, the problem is even more complicated. Furthermore, water, the solvent most relevant to biology, is a particularly complicated liquid, a point that we will return to later in the course. So, if we were to make up a solution by mixing a given amount of solute with a given amount of solvent, we would probably have to measure the final volume in order to relate volume to the amount of solvent. (For a few, particularly well-characterized solute-solvent pairs, very precise measurements have been made that can be used to predict volumes and densities of solutions.)

There are some special applications for which it is advantageous to use units of concentrations based on the amounts of solute and solvent, rather than total volume. In particular, molal units are used often in chemical thermodynamics. A 1 molal solution is prepared by dissolving 1 mole of solute in 1 kg of solvent. The advantage of a concentration defined this way is that it does not change with temperature or pressure; the masses of solute and solvent remain the same. In contrast, a change in temperature or pressure can change the total volume of a solution and, therefore the molar concentration. The behaviors of the two solutions aren't really different, it's just that the definition of a molar concentration depends on volume, and the molal concentration doesn't. As in many things, it is a matter of what is most important to keep track of for a particular purpose.

Now, it should also be noted that the practical difference between solutions defined by the amount of solvent versus a given total volume depends greatly on just how concentrated the solutions are. If, for instance, the solute makes up less than 1% of the total volume of the solution, just mixing the solute with, say, 1 L of solvent may not introduce a significant error for many purposes. In biochemistry labs, solutions are often made up to quite small volumes (usually because the reagents are very expensive), and this kind of error is frequently considered acceptable. On the other hand, the solute may make up as much as half of the total volume of some solutions, and the difference between, say, a 5 M solution and a 5 molal solution is very significant, indeed.

In some situations, solution concentrations are expressed as percentages, and here there is also an important distinction. Percentage solutions can be specified as either mass per volume or volume per volume. A percent mass/volume,%(m/v), concentration, sometimes identified as %(w/v), is defined as the number of grams of solute dissolved in enough solute to make 100 mL total volume. As with molar concentrations, it is easy to calculate the amount of solute in a given volume of solution for a %(m/v) solution. Percent volume/volume, %(v/v), concentrations are usually used for solutions made by mixing volumes and are defined by the number of mL of one liquid mixed with a second liquid to give a final volume of 100 mL. Depending on the densities of the liquids, there may be a significant difference between the percent concentrations expressed as m/v and v/v for the same solution.

1.4.2 Units of atomic and molecular mass

A variety of terms are used to describe the masses of atoms, ions and molecules, and there is a bit of confusion and inconsistency in their definitions and use. At first glance, it might seem that atomic and molecular masses should be expressed in the usual SI unit of mass, the kilogram. But, the mass in kilograms of a single atom or molecule is a very small and awkward number for most uses. So, instead we have a special unit, which goes by different names in different contexts. Both of the following refer to the same unit:

- Unified atomic mass unit, abbreviated as u or amu
- dalton, abbreviated as Da

Niether of these equivalent units is defined in the SI, but are defined by the International Union of Pure and Applied Chemistry (IUPAC). The term amu is widely used in the field of mass spectrometry (where molecular masses are sometimes measured with precision of a fraction of an amu), whereas the dalton is more widely used in the molecular biosciences.

These two equivalent units are defined as the mass of an atom, ion or molecule divided by the mass of an atom of carbon 12 (^{12}C) divided by 12. By this definition, then, an atom of ^{12}C has a mass of exactly 12 Da. 1 amu, or 1 da is approximately equal to 1.66054×10^{-27} kg.

Although we tend to think of the atomic mass of an atom as being an integer, equal to the total number of protons and neutrons in the nucleus, the masses of the protons and neutrons do not add exactly, because of the presence of other subatomic particles. So, the masses of atoms other than ^{12}C generally differ slightly from an integer. In addition, when a sample of an atomic or molecular species is considered, there are usually more than a single isotope present, with different masses, so that the average atomic or molecular mass of the species usually deviates significantly from an integer. The molecular masses that are usually cited in articles, books and on the labels on bottles of chemicals are based on the average ratios of the various isotopes found in nature (on our planet). But, these ratios can differ slightly for natural reasons and can be altered greatly by artificial enrichment.

Because the amu and dalton are defined as the ratio of two masses, they are really units without dimensions, and is common and legitimate to represent atomic and molecular masses as pure numbers without units. To emphasize the relative nature of atomic and molecular masses, IUPAC defines the terms *relative atomic mass* and *relative molecular mass*, with the symbols A_r and M_r, respectively. The terms *atomic weight* and *molecular weight* are also deeply ingrained in many of us, but aren't strictly correct, since weight is a measure of (gravitational) force, rather than mass.

Two other commonly used terms are *molar mass* and *relative molar mass*, both of which refer to the mass of one mole of a substance[6]. Logically, molar mass would simply have the units of g or kg, but it is usually expressed as g/mol, which is a bit redundant. The

[6] The definition of the molar mass has been muddied somewhat by the redefinition of the mole in 2019, but not in a way that is likely any to have any practical differences. Prior to 2019, the mole was defined as the number of ^{12}C atoms in exactly 12 g. This meant that, by definition, the molar mass of ^{12}C was exactly 12 g/mol, and the molar masses of other entities were similarly linked exactly to their atomic or molecular masses. With the mole now redefined as exactly 6.02214076×10^{23} particles, the direct connection to the atomic mass and molar mass of ^{12}C is broken. But, any discrepancy is on the order of parts per billion!

relative molar mass is defined by IUPAC as the molar mass divided by 1 g/mol, making it dimensionless and equal to the relative molecular mass, M_r.

So, we have the following terms (for molecules) which all have the same numerical values and convey the same information:

- Molecular mass (or weight), with units of daltons (da) or unified atomic mass unit (u or amu).
- Relative molecular mass (or weight), without units and represented as M_r.
- Molar mass, with units g/mol.
- Relative molar mass, without units and equivalent to relative molecular mass.

These distinctions are all rather picky, but for clarity one should be careful to use the appropriate units when using one of these terms. The important point, for practical purposes is this: Given the molecular mass in any of these forms, one has in hand the conversion factor for converting between mass in grams and the number of moles. If we have a molecular species with a relative molecular mass, M_r, we can write:

$$1\,\mathrm{mole} = M_\mathrm{r}\mathrm{g}$$
$$M_\mathrm{r}\mathrm{g/mol} = 1$$

So, for instance, if we have 30.0 of glucose with a molecular mass of 180.16 Da, we can calculate the number of moles as:

$$30.0\,\mathrm{g} \div 180.16\,\mathrm{g/mol} = 0.166\,\mathrm{mol}$$

1.4.3 Special units of concentration for hydrogen and hydroxide ions

There are two ionic species that receive special attention whenever water is involved, and these are the hydrogen (H^+) and hydroxide (OH^-) ions. These two are always present, though usually at quite low concentrations, in aqueous solutions because water itself has a tendency to dissociate to produce both:

$$H_2O \rightleftharpoons H^+ + OH^-$$

The forward dissociation reaction rate is actually quite slow, so that the average time for a given water molecule to dissociate is about 11 h. However, even a small volume of water contains a large number of water molecules, and re-association of H^+ and OH^- is very fast, occurring essentially instantaneously once the the two ions colide in solution. As a consequence the forward and reverse reactions reach a balance in a fraction of a second. We will discuss equilibrium constants in more depth later in the course, when we study thermodynamics, but for now it is sufficient to say that the reaction quickly reaches an equilibrium state such that the concentrations of H^+ and OH^- are related to one another according to:

$$[\mathrm{H}^+]_\mathrm{eq}[\mathrm{OH}^-]_\mathrm{eq} = 10^{-14}\,\mathrm{M}^2$$

where $[H^+]_{eq}$ and $[OH^-]_{eq}$ are the equilibrium concentrations of the two ions. Because the dissociation reaction reaches equilibrium very rapidly, we generally assume that the concentrations of H^+ and OH^- in a solution satisfy the equilibrium condition. An important consequence of this relationship is that if the concentration of either H^+ or OH^- is known, the concentration of the other is also determined. In an absolutely pure sample of water, the dissociation reaction should be the only source of H^+ and OH^-, and their concentrations should be equal. This defines what we describe as a neutral solution, and the equilibrium expression is satisfied under these conditions when the concentrations of H^+ and OH^- are 10^{-7} M.

Because the concentration of H^+ and OH^- can vary over a wide range, it is convenient to use special representation for their concentration, pH and pOH, respectively. The pH and pOH of a solution are defined by"

$$\mathrm{pH} = -\log [\mathrm{H}^+]$$
$$\mathrm{pOH} = -\log [\mathrm{OH}^-]$$

From this definition and the discussion above, you should be able to readily demonstrate that the pH and pOH of a neutral aqueous solution are both equal to 7 and that the sum of pH and pOH is equal to 14 for any solution. Although either pH or pOH can be used to describe the equilibrium concentrations of H^+ and OH^- in a solution, pH is, by far, the more commonly used parameter.

Earlier, we estimated the volume of an *E. coli* bacterium to be about 10^{-15} L. An interesting implication of this very small volume is that the number of some molecules and ions in a single cell are surprisingly small. For instance, we can ask: How many hydrogen ions are in a bacterium? If the pH in the cell is 7, then the concentration is

$$[\mathrm{H}^+] = 10^{-\mathrm{pH}}\mathrm{M} = 10^{-7}\,\mathrm{M} = 10^{-7}\,\mathrm{moles/L}$$

The number of moles of H^+ ions is then calculated as

$$10^{-7}\,\mathrm{moles/L} \times 1.6 \times 10^{-15}\,\mathrm{L} = 1.6 \times 10^{-22}\,\mathrm{moles}$$

To calculate the number of ions, the number of moles is multiplied by Avogadro's number, which can be thought of as having the units of particles/mole

$$1.6 \times 10^{-22}\,\mathrm{moles} \times 6.02 \times 10^{23}\,\mathrm{ions/mole} \approx 100\,\mathrm{ions}$$

That's not very many!

There are also bacteria that grow at pH 9. How many hydrogen ions are present in one of these bacteria?

1.5 Further reading

For an authoritative reference on the SI units and conversion factors, see:

- Thompson, A. & Taylor, B. N. (2008). Use of the international system of units (SI). NIST Special Publication 811, National Institute of Standards and Technology, Gaithersburg, MD.
  http://physics.nist.gov/cuu/Units/bibliography.html

A convenient online unit conversion tool:

- http://www.digitaldutch.com/unitconverter/
  Keep this disclaimer in mind: "While we try really hard to make all calculations accurate, we do not guarantee that the results you get are correct. If you do find any bugs I highly appreciate it if you email us at info@digitaldutch.com"

Wikipedia contains a number of good articles on the metric system, including its history and the current SI. As a publicly edited secondary source, some caution is always advised when using Wikipedia (or any source, really), but these articles are well referenced, so that the primary sources can be checked.

- https://en.wikipedia.org/wiki/Metric_system
- https://en.wikipedia.org/wiki/Metre
- https://en.wikipedia.org/wiki/Kilogram
- https://en.wikipedia.org/wiki/International_System_of_Units
- https://en.wikipedia.org/wiki/2019_redefinition_of_the_SI_base_units

Wikipedia articles on English measurement systems:

- https://en.wikipedia.org/wiki/English_units
- https://en.wikipedia.org/wiki/Imperial_units
- https://en.wikipedia.org/wiki/United_States_customary_units

Articles on the redefinition of the kilogram and mole with new technologies:

- Cho, A. (2018). World poised to adopt new metric units. *Science*, 362, 625–626.
  http://doi.org/10.1126/science.362.6415.625

- Cho, A. (2017). Plot to redefine the kilogram nears climax. *Science*, 356, 670–671.
  http://dx.doi.org/10.1126/science.356.6339.670

- Robinson, I. A. & Schlamminger, S. (2016). The watt or Kibble balance: a technique for implementing the new SI definition of the unit of mass. *Metrologia*, 53, A46.
  http://dx.doi.org/10.1088/0026-1394/53/5/A46

- https://www.nist.gov/physical-measurement-laboratory/silicon-spheres-and-international-avogadro-project