December 30, 2024 Draft

Physical Principles in Biology

Biology 3550/3551

Spring 2025

Chapter 3: Random Walks

David P. Goldenberg
University of Utah
goldenberg@biology.utah.edu

Chapter 3

Random Walks

We have now spent a lot of time looking at "plinko probabilities", and you should have a good feel for how bell curves arise and how to calculate the probabilities of different outcomes in a binomial distribution. Now, we want to start talking about random walks and the ways in which they arise in physical and biological contexts.

Although the binomial distribution can, in principle, be used to describe a random walk in one dimension, actually using this function for large numbers of steps quickly becomes problematic. Calculating $n!$ requires $n-1$ multiplications, and the magnitudes of the numbers quickly become difficult to handle.. Also, we need to move beyond one dimension. So, we need some other mathematical approaches.

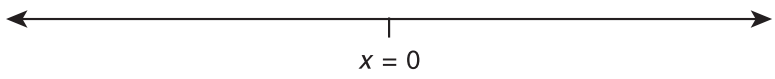
We will start, however, by considering a one-dimensional random walk.

3.1 Random walks in one dimension

In the simplest version of a random walk, we imagine an individual standing on a sidewalk and flipping a coin. If the coin lands heads-up, she turns in one direction and takes a step of length l. If the coin lands tails-up, she turns the opposite direction and takes a step of length l. She then repeats this process another $n-1$ times, for a total of n steps in the random walk.

I. The final position of the walker.

We will define the walker's position as x, which is zero at the beginning of the random walk. As the walker takes steps in the opposite directions, the value of x can take on positive or negative values, as illustrated by the single coordinate axis drawn below:



We will call the position after i steps, x_i, and the final position, after n steps, is x_n. At the outset, we can assume a few things abut the value of x_n, irrespective of whether the coin is fair or not:

- The maximum possible value of x_n is nl
- The minimum possible value of x_n is $-nl$
- Assuming that n is very large, the probability of a walk ending at either nl or $-nl$ is very small, since either outcome would require the coin to land the same way for each toss.

- If a large number of random walks are carried out the distribution of x_n should be related to a binomial distribution.

As a first step in analyzing the random walk in one dimension, we will calculate the expected value of x_n, that is the expected average value of x_n if a large number of random walks, each of n steps, is executed. Here and through out the discussion of random walks, we will call the number of steps in an individual random walk n, and the number of random walks, as used for calculating averages, N.

For each random walk, the final position is given by:

$$x_n = \sum_{i=1}^{n} \delta_i$$

where i is the step number, and δ_i is the change in x in step i. If the step is to the right, $\delta_i = l$, whereas if the step is to the left, $\delta_i = -l$. We will call the probability of an individual step to the right p_+ and the probability of a step to the left p_-.

The expected value of δ_i, for any individual step, is calculated as:

$$\begin{aligned} E(\delta_i) &= lp_+ - lp_- \\ &= lp_+ - l(1 - p_+) \\ &= lp_+ + lp_+ - l \\ &= 2lp_+ - l = l\big(2p_+ - 1\big) \end{aligned}$$

As a quick check, note that if the probability of left and right steps are equal, $p_+ = 0.5$, and the expected value of δ_i is 0.

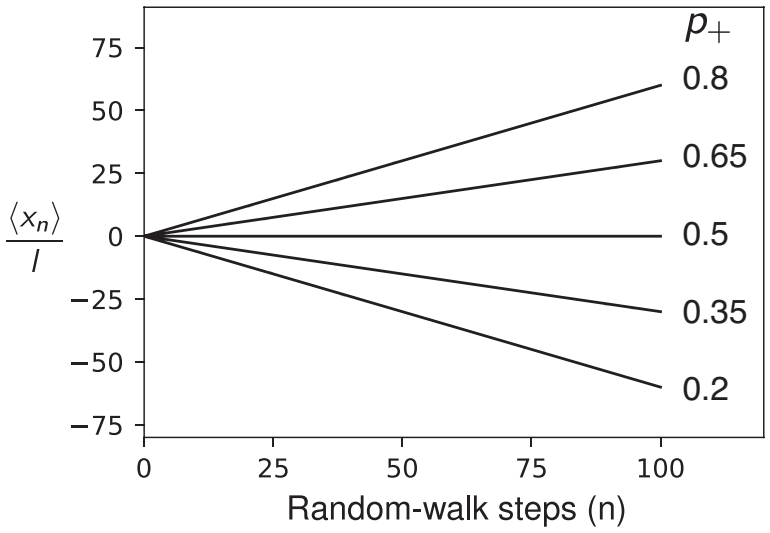
An important theorem from probability states that if x and y are two independent random variables, then the expected value of the sum of x and y is calculated as:

$$E(x + y) = E(x) + E(y)$$

Since x_n is simply the sum of δ_i for each step in the random walk, the expected value of x_n is calculated as:

$$\begin{aligned} E\big(x_n\big) &= \sum_{i=1}^{n} E(\delta_i) \\ &= \sum_{i=1}^{n} l\big(2p_+ - 1\big) \\ &= nl\big(2p_+ - 1\big) \end{aligned}$$

Note that if $p_+ = 0.5$, then the expected value of x_n is zero, that is the average final position is the starting point, irrespective of the number of steps. The plot below shows the expected value of x_n as a function of n for different probabilities of an individual forward step.



As one might expect, values of p_+ greater than 0.5 favor positive values of x_n, and values of p_+ less than 0.5 favor negative values of x_n. Notice also that, for any given value of p_+ (except 0.5), the expected value of x_n increases or decreases linearly with the number of steps.

If the number of individual random walks, N, is large, then the average value of x_n will approach the expected value. Note the distinction between the average value of x_n for N specific random walks and the expected value, $E\big(x_n\big)$ which is calculated from the probabilities of individual forward and reverse steps, as well as the number of steps. Mathematically, we would write the relationship between the two as:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} x_{n,j} = E\big(x_n\big)$$

where the index j indicates the individual random walks included in the average.

To write averages over multiple random walks in a more compact form, we will employ the practice of representing averages using pairs of angle brackets, $\langle\rangle$, as in the example:

$$\langle x_n \rangle = \frac{1}{N} \sum_{j=1}^{N} x_{n,j}$$

Though it is a bit sloppy, we will generally take averages represented in this way to mean that N is large enough that the average approaches the expected value, unless N is otherwise specified.

II. Other averages: The mean-square and root-mean-square

As shown above, it is quite easy to calculate the expected value of x_n for a one-dimensional random walk, even when the probabilities of turns in the two directions are not equal. However, this average provides only limited information. From our study of plinkos, we know that most of the balls don't actually land in the central

bucket (or central two buckets when the number of rows is odd), even though landing in the central bucket(s) is the most probable result.

What we need as a way to represent the distribution of final positions away from the average value of x_n. For this purpose, there are two other kinds of average, which are widely used in a variety of contexts. These are the mean-square and root-mean-square averages, which are defined below, using the angle brackets to represent averages.

The mean-square:

$$\langle x_n^2 \rangle = \frac{1}{N} \sum_{j=1}^{N} x_{n,j}^2$$

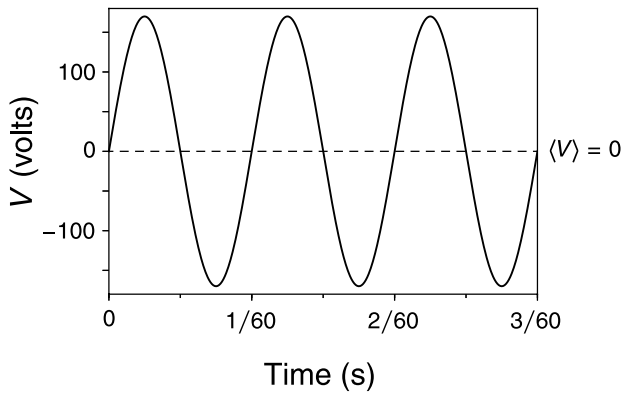
where, as before, the sum is over the N random walks.

The root-mean-square (RMS):

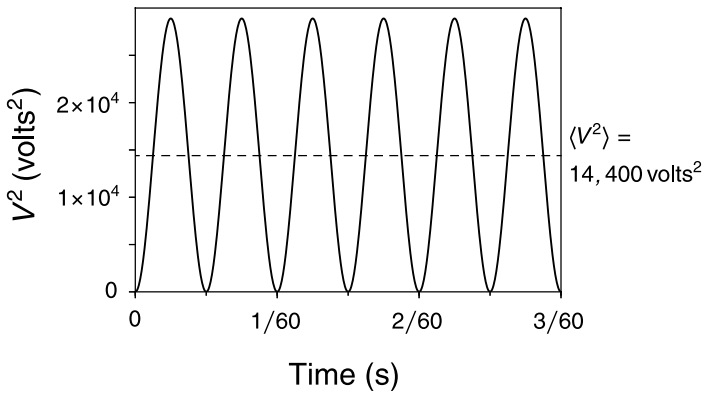
$$\text{RMS}(x_n) = \sqrt{\langle x_n^2 \rangle} = \sqrt{\frac{1}{N} \sum_{j=1}^{N} x_{n,j}^2}$$

By summing over the squares of the final positions, both positive and negative values of x_n make a positive contribution to the averages, rather than canceling out, as when the simple average of x_n is calculated. This goal could be also be obtained by using the absolute value of x_n, but absolute values are more awkward when deriving general results, and the squared quantities have important statistical significance.

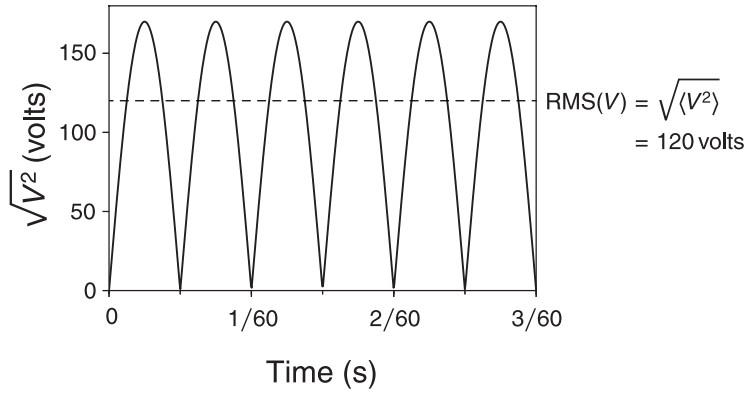
A common application of the mean-square and RMS averages is in electrical engineering, where they are used to treat alternating currents (AC). The graph below shows ideal behavior of voltage as a function of time, for the AC power used in US homes.



Note that the voltage oscillates between a maximum of 170 V and a minimum of −170 V, with the average over time being 0 V. For the US power system, each cycle takes 1/60 s, for a frequency of 60 Hz. Although mathematically correct, the simple average of the voltage over time obviously doesn't convey much information about the power available frome the current. To obtain a positive value, the instantaneous voltage can be squared to generate the plot below.



In this plot, both the maxima and minima in the original plots give rise to peaks of $28900\,\mathrm{V}^2$. The average square voltage over time is $14400\,\mathrm{V}^2$. Although this average is positive and reflects the magnitude of both the positive and negative voltage fluctuations, it has the disadvantage of being expressed in units of V^2, which are not so intuitively interpreted. This is the main reason for introducing the root-mean-square (RMS) average. In the plot below, the instantaneous voltage vales are squared, and the the square root is taken for each point.



Notice that the peaks in the plot have slightly different shapes than those in the plot of V^2, and the peaks have a height of 170 V. The RMS average over time is 120 V, and it is this average that is usually specified for AC circuits. Note that the RMS average is calculated as the square root of the mean-square average and *not* by averaging over the square root of the squares of the individual values, which would, in general, give a different result.

III. The mean-square and RMS end-to-end distance of a one-dimensional random walk

For the reasons discussed above, we would like to have an average that represents distance between the beginning and end of a random walk, calculated in a way that positive and negative steps don't cancel one another. We will begin with the mean-square distance, $\langle x_n^2 \rangle$, which is easier to work with. Once an expression for $\langle x_n^2 \rangle$ is derived, the root-mean-square is calculated by taking the square root.

We start with a definition of the mean-square distance for the random walk:

$$\langle x_n^2 \rangle = \frac{1}{N} \sum_{j=1}^{N} x_{n,j}^2$$

where N is the number of random walks, and the index j represents the individual random walks. For each of the random walks, the final position is given by:

$$x_n = \sum_{i=1}^{n} \delta_i$$

where δ_i is the change in position along the x-axis and can be either $+\delta$ or $-\delta$. Though the reason for doing so may not be obvious yet, we can also write x_n as:

$$x_n = x_{n-1} + \delta_n$$

where δ_n is the change in x in the very last step of the walk. Using this representation, the mean-square distance can be written as:

$$\begin{aligned}
\langle x_n^2 \rangle &= \frac{1}{N} \sum_{j=1}^{N} x_{n,j}^2 \\
&= \frac{1}{N} \sum_{j=1}^{N} \Big(x_{(n-1),j} + \delta_{j,n} \Big)^2 \\
&= \frac{1}{N} \sum_{j=1}^{N} \Big(x_{(n-1),j}^2 + 2 x_{(n-1),j} \delta_{j,n} + \delta_{j,n}^2 \Big)
\end{aligned}$$

where $\delta_{j,n}$ is the change in x for the last step in the j^{th} random walk. This can be broken down into individual sums and averages to give:

$$\begin{aligned}
\langle x_n^2 \rangle &= \frac{1}{N} \sum_{j=1}^{N} x_{(n-1),j}^2 + \frac{1}{N} \sum_{j=1}^{N} 2 x_{(n-1),j} \delta_{j,n} + \frac{1}{N} \sum_{j=1}^{N} \delta_{j,n}^2 \\
&= \langle x_{n-1}^2 \rangle + \langle 2 x_{n-1} \delta_n \rangle + \langle \delta_n^2 \rangle
\end{aligned}$$

As before the angle brackets represent averages over a large number of random walks. For each random walk, the final change in x will be either l or $-l$ and will be uncorrelated with the position, x_{n-1}. If we limit ourselves to the case where the probability of a forward or backward step is equal, the central term in the expression above, $\langle 2 x_{n-1} \delta_n \rangle$, will be zero. Thus, we can write:

$$\langle x_n^2 \rangle = \langle x_{n-1}^2 \rangle + \langle \delta_n^2 \rangle$$

Note that the average of δ_n^2 over all of the random walks is *not* expected to be zero.

Following the same arguments as above, the position of the walker after $n-1$ steps can ve written as:

$$x_{n-1} = x_{n-2} + \delta_{n-1}$$

and the average of x_{n-1} is:

$$\langle x_{n-1}^2 \rangle = \langle x_{n-2}^2 \rangle + \langle \delta_{n-1}^2 \rangle$$

The mean-square average of x_n can then be written as:

$$\begin{aligned} \langle x_n^2 \rangle &= \langle x_{n-1}^2 \rangle + \langle \delta_n^2 \rangle \\ &= \langle x_{n-2}^2 \rangle + \langle \delta_{n-1}^2 \rangle + \langle \delta_n^2 \rangle \end{aligned}$$

Since the individual steps in a random walk are uncorrelated, and the individual walks are uncorrelated, the average values of δ_{n-1}^2 and δ_n^2 should be the same, so that we have:

$$\langle x_n^2 \rangle = \langle x_{n-2}^2 \rangle + 2\langle \delta_i^2 \rangle$$

where $\langle \delta_i^2 \rangle$ is the mean-square average of the change in x, averaged over all of the steps in the random walks.

The same logic can be applied repeatedly:

$$\begin{aligned} \langle x_n^2 \rangle &= \langle x_{n-2}^2 \rangle + 2\langle \delta_i^2 \rangle \\ &= \langle x_{n-3}^2 \rangle + \langle \delta_{n-2}^2 \rangle + 2\langle \delta_i^2 \rangle \\ &= \langle x_{n-3}^2 \rangle + 3\langle \delta_i^2 \rangle \\ &= \langle x_{n-4}^2 \rangle + 4\langle \delta_i^2 \rangle \end{aligned}$$

and so on, until we have:

$$\begin{aligned} \langle x_n^2 \rangle &= \langle x(1)^2 \rangle + (n-1)\langle \delta_i^2 \rangle \\ &= \langle x(0)^2 \rangle + n\langle \delta_i^2 \rangle \\ &= n\langle \delta_i^2 \rangle \end{aligned}$$

This derivation does not depend on any assumptions about the value of $\langle \delta_i^2 \rangle$, though it does assume that $\langle \delta_i \rangle$ is zero. If we further assume that δ_i is either l or $-l$, with equal probability, then the average of δ_i^2 can be further specified from the expected value:

$$\begin{aligned} \langle \delta_i^2 \rangle = E(\delta_i^2) &= p_+ l^2 + p_-(-l)^2 \\ &= p_+ l^2 + p_- l^2 \\ &= l^2\big(p_+ + p_-\big) \\ &= l^2 \end{aligned}$$

We can then write $\langle x_n^2 \rangle$ in the terms defining the random walk, n, the number of steps and l, the length of each step:

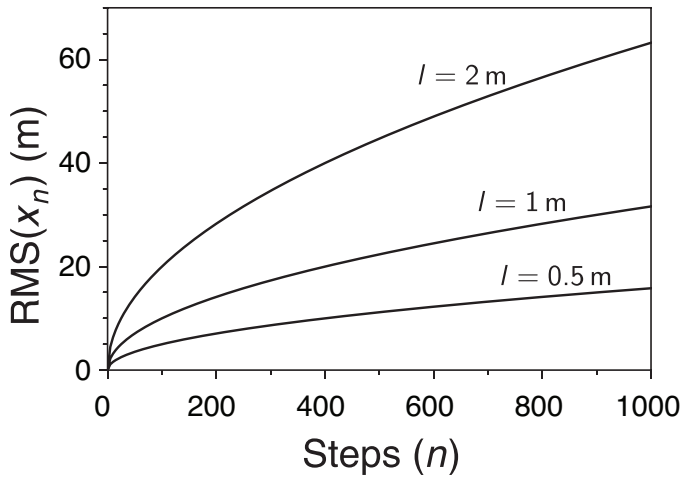
$$\langle x_n^2 \rangle = nl^2$$

The root-mean-square distance between the starting and ending positions is then given by:

$$\mathrm{RMS}(x_n) = \sqrt{nl^2} = \sqrt{n}l$$

Note that $\mathrm{RMS}(x_n)$ has the same dimensions, length, as the step length, l.

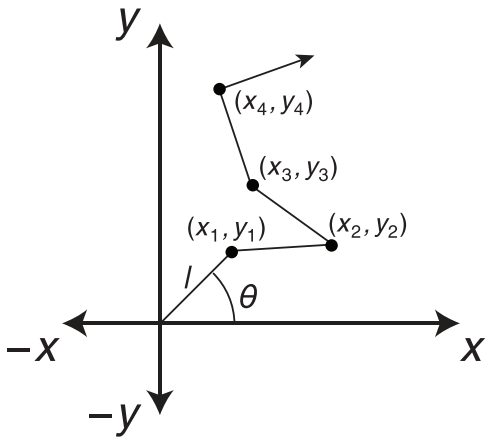
This is the key result: The RMS average distance increases with the square root of the number of steps. It doesn't increase linearly with the number of steps, because not every step moves the walker away from the starting point. But, the average distance isn't zero, even though the average position, $\langle x_n \rangle$ is zero. The relationship between the RMS end-to-end distance and the number of steps is shown in the figure below:



Note that for small values of n the RMS distance increases relatively rapidly with n. This is because, for a small number of coin flips, for instance, there is a relatively large probability that a significant majority will be either heads or tails. However, as the number of coin flips, n, increases, the likelihood of a significant deviation from the expected average decreases, and the RMS distance increases with n more slowly. For any given number of steps, $\mathrm{RMS}(x_n)$ is proportional to the step length.

3.2 Random walks in two dimensions

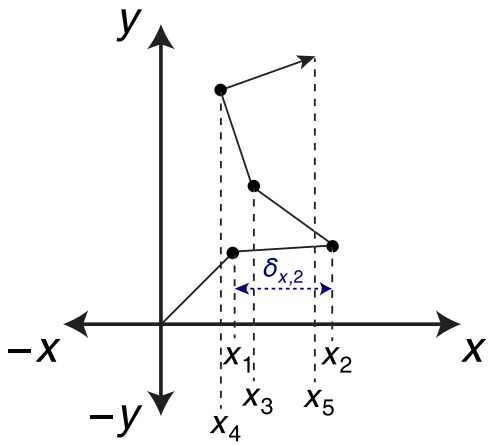
In the simplest form of a two-dimensional random walk, a walker begins at the origin of a two-dimensional coordinate system, where $x = 0$ and $y = 0$, and chooses at random an angle, θ, between 0 and 2π rad. The walker then takes a step, of length l in the direction defined by the angle θ with respect to the x-axis, as diagrammed below:



The process is then repeated $n-1$ times to generagte an n-step random walk.

I. The random walks along the x- and y-axes

As the walker generates a path in two dimensions, it also can be thought of as carrying out a walk along the x-axis. With each step, the projection of the current walker position onto the x-axis changes, as illustrated below:



For each step, the change in the x-coordinate is

$$\delta_{x,i} = x_i - x_{i-1}$$

Just as we did for the one-dimensional random walk along the x-axis, we can calculate the following averages for the walk defined by the projections along the x-axis:

$$\langle x_n \rangle = \frac{1}{N} \sum_{j=1}^{N} x_{n,j}$$

$$\langle x_n^2 \rangle = \frac{1}{N} \sum_{j=1}^{N} x_{n,j}^2$$

$$\text{RMS}(x_n) = \sqrt{\langle x_n^2 \rangle}$$

The central assumption that we will make at this point is that the turn angle at each step is equally likely to take on any value between 0 and 2π rad. This means that positive and negative changes in the x-coordinate are equally likely, leading to the result:

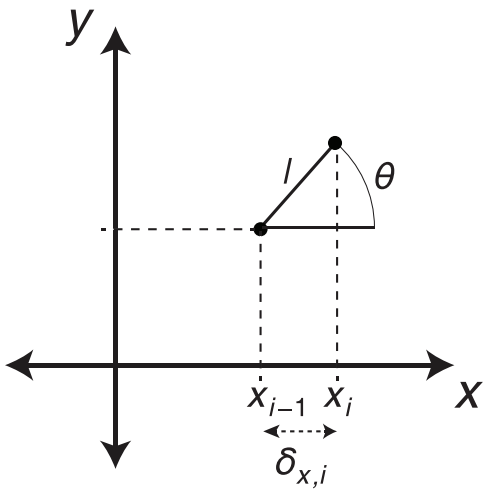
$$\langle x_n \rangle = 0$$

Recall that for the one-dimensional random walk we showed that:

$$\langle x_n^2 \rangle = n\langle \delta^2 \rangle$$

where $\langle \delta^2 \rangle$ is the mean-square average of the changes in position along the x-axis. This derivation assumed only that $\langle \delta_i \rangle = 0$, so it applies to the case of the x-projections in the two-dimensional random walk, as well.

For the one-dimensional random walk, we also argued that the individual changes in the x-position could only be l and $-l$ and, therefore, $\langle \delta_i^2 \rangle = l^2$. However, this argument does not apply to the changes in the x-projections in the two-dimensional random walk. To see why, consider the change in x-coordinate for a single step, as diagrammed below:



If the angle, θ is zero, then $\delta_{x,i} = l$, and $\delta_{x,i}^2 = \delta^2$. If θ is π, then $\delta_{x,i} = -\delta$, and $\delta_{x,i}^2 = l^2$. However, for most values of θ, $\delta_{x,i}$ lies between $-l$ and l and $\delta_{x,i}^2$ is less than l^2

To calculate the average value of $\delta_{x,i}^2$, we calculate the expected value for a continuous probability distribution function (see page 60):

$$\langle \delta_{x,i}^2 \rangle = E(\delta_{x,i}^2) = \int_{-\delta}^{\delta} \delta_x^2 p(\delta_x) d\delta_x$$

It's not so obvious what the probability distribution function, $p(\delta_x)$ is, but the random variable δ_x is related to the random variable θ according to:

$$\delta_x = l \cos\theta$$

From this relationship, the expected value of $\delta_{x,i}^2$, $\langle\delta_{x,i}^2\rangle$, can be calculated by integration with respect to θ:

$$\begin{aligned}\langle\delta_{x,i}^2\rangle &= \int_0^{2\pi} \delta_x(\theta)p(\theta)d\theta \\ &= \int_0^{2\pi} (l\cos\theta)^2 p(\theta)d\theta\end{aligned}$$

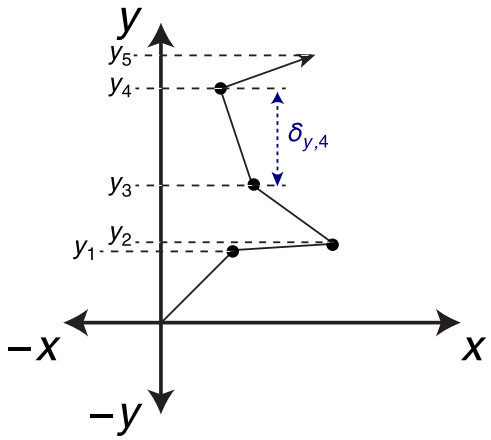
In Chapter 2 (page 59), it was shown that $p(\theta) = 1/(2\pi)$ for a uniform distribution of θ between 0 and 2π. The integral can then be evaluated as:

$$\begin{aligned}\langle\delta_x^2\rangle &= \frac{1}{2\pi}\int_0^{2\pi} (l\cos\theta)^2 d\theta \\ &= \frac{l^2}{4\pi}\left(\frac{\sin(2\theta)}{2} + \theta\right)\Bigg|_{\theta=0}^{2\pi} \\ &= \frac{l^2}{2}\end{aligned}$$

The mean-square projection along the x-axis of the endpoint after n steps is then calculated as:

$$\begin{aligned}\langle x^2\rangle &= n\langle\delta_x^2\rangle \\ &= \frac{nl^2}{2}\end{aligned}$$

The two-dimensional random walk can also be envisioned as creating a random walk along the y-axis:



There is nothing really special about either the x- or y-axis, or any other direction (though the *relationship* between the x- and y-axis is special, because they are perpendicular to each other). As a consequence, the results derived for the averages of the

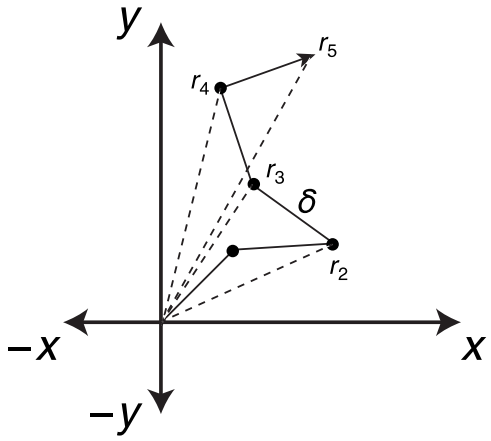
projections along the x-axis can be directly applied to the y-axis:

$$\langle y \rangle = 0$$

$$\langle y^2 \rangle = \frac{nl^2}{2}$$

II. The end-to-end distance

In addition to the projections along the x- and y-axis for a two-dimensional random walk, we can consider the distance between the starting and ending positions along the straight line connecting them, as opposed to the actual path of the walk. The diagram below shows how the distance of the walker from the starting position, r_i, changes as the number of steps in the random walk increases:



At the end of any specific random walk, the distance from the starting point, r_n, is related to the x- and y-projections according to:

$$r_n = \sqrt{x_n^2 + y_n^2}$$

and

$$r_n^2 = x_n^2 + y_n^2$$

To calculate the mean-square end-to-end distance, we again use the theorem for the expected value of a sum of two random variables. For two random variables, A and B, with expected values $E(A)$ and $E(B)$:

$$E(A + B) = E(A) + E(B)$$

The expected value for r^2 can thus be written:

$$E(r_n^2) = E(x_n^2) + E(y_n^2)$$

Assuming, as we have, that the number of random walks, N, over which the averages are taken is very large, this can be expressed in terms of the mean-square averages:

$$\langle r_n^2 \rangle = \langle x_n^2 \rangle + \langle y_n^2 \rangle$$

In the previous section, we showed that

$$\langle x_n^2 \rangle = \langle y_n^2 \rangle = nl^2/2$$

By substitution, we have:

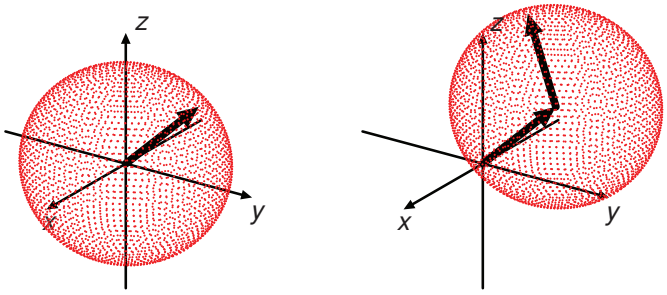
$$\begin{aligned} \langle r_n^2 \rangle &= \langle x_n^2 \rangle + \langle y_n^2 \rangle \\ &= nl^2/2 + nl^2/2 \\ &= nl^2 \end{aligned}$$

Thus, we have exactly the same result as for the one-dimensional random walk! The root-mean-square end-to-end distance is also the same as for the one-dimensional case:

$$\text{RMS}(r_n) = \sqrt{\langle r^2 \rangle} = \sqrt{n}l$$

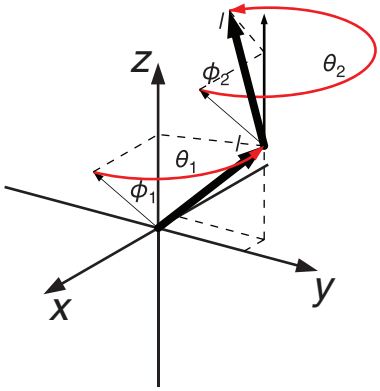
3.3 Three-dimensional Random Walks

A random walk in three-dimensions can be represented as a series of vectors in a three-dimensional coordinate system. The first step begins at the origin, as shown in the left-hand panel below, and ends on a random point on the surface of a sphere with its center at the origin and a radius equaling the step length.



The second step begins the end of the first and ends on a point on the sphere with its center at the starting point for the step, as illustrated in the right-hand panel.

To describe the changes in direction for each step, it is useful to use polar coordinates, as illustrated below:



In the polar-coordinate system, the position of the endpoint of a vector is described by the length of the vector and two angles. The vector is visualized as beginning initially aligned with the z-axis and then being rotated by an angle, ϕ, away from the z-axis in the plane of the x-z plane, and then rotated by an angle, θ, about the z-axis.

We can derive an expression for the mean-square end-to-end distance for a three-dimensional random walk by following the same general approach as for the two-dimensional case. For that case, we showed that

$$\langle r^2 \rangle = \langle x^2 \rangle + \langle y^2 \rangle$$

where $\langle x^2 \rangle$ and $\langle y^2 \rangle$ are the mean-square projections of the random-walk end-points onto the x- and $y-$ axes, respectively. For the three-dimensional case:

$$\langle r^2 \rangle = \langle x^2 \rangle + \langle y^2 \rangle + \langle z^2 \rangle$$

In order to calculate $\langle x^2 \rangle$, $\langle y^2 \rangle$ and $\langle z^2 \rangle$, we need to consider the distributions of the projections of the individual steps onto the three axes and then calculate $\langle \delta_{x,i} \rangle$, $\langle \delta_{y,i} \rangle$ and $\langle \delta_{z,i} \rangle$.

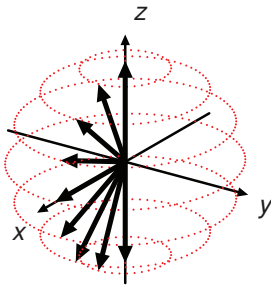
If the direction of each step is random with respect to the coordinate axis, the mean-square projection along any direction is the same as along any other direction. When using the polar coordinate system as defined above, the z-axis is the most convenient to work with, since the projection for a single step depends only on the step length, l, and the angle ϕ:

$$\delta_z = l \cos \phi$$

The mean-square projection of an individual step onto the z-axis is given by:

$$\langle \delta_z^2 \rangle = \int_0^\pi p(\phi) \delta_z^2 d\phi = \int_0^\pi p(\phi) (l \cos \phi)^2 d\phi$$

To evaluate this expression, we need to know the probability distribution function for the angle ϕ, $p(\phi)$. At first glance it might seem that all values of ϕ would be equally probable, so that $p(\phi)$ would be a simple constant. Recall, however, that the steps in the random walk were defined so that a step towards any point in the surrounding sphere is equally probable. The figure below represents the effect of rotating the vector by different values of ϕ from the z-axis and the points on the sphere that are accessible as the vector is rotated about the z-axis.



As indicated in the figure, the largest number of points on the sphere is associated with a rotation that places the vector in the x-y plane, corresponding to a value of ϕ equal to

$\pi/2$. At the other extreme, if $\phi = 0$ or π, the number of points is infinitesimally small. More generally, the number of points accessible for a given value of ϕ is proportional to the circumference of the circle swept out by the vector as it rotates about the z-axis. The circumference in turn is proportional to the radius, r_c, which is related to ϕ according to:

$$r_c = l \sin\phi$$

In order to satisfy the requirement that all directions in three dimensions be equally probable, the probability distribution function for ϕ must be proportional to $\sin\phi$.[1] The probability distribution function can thus be written in the form of:

$$p(\phi) = c \sin\phi$$

where c is a constant of proportionality. To evaluate this constant, we impose the requirement that the distribution function must be normalized:

$$\begin{aligned}\int_0^\pi p(\phi)d\phi &= \int_0^\pi c\sin\phi d\phi = 1 \\ &= \big(-c\cos\phi\big)\Big|_0^\pi \\ &= c - (-c) = 2c\end{aligned}$$

The constant c must then be equal to $1/2$ in order for the probability density function to be normalized:

$$p(\phi) = \frac{1}{2}\sin\phi$$

The average, or expected, value of the step-length projection onto the z-axis is then:

$$\begin{aligned}\langle\delta_z^2\rangle &= \int_0^\pi p(\phi)\delta_z^2 d\phi = \int_0^\pi p(\phi)(l\cos\phi)^2 d\phi \\ &= \int_0^\pi \frac{1}{2}\sin\phi(l\cos\phi)^2 d\phi \\ &= \frac{l^2}{2}\int_0^\pi \sin\phi\cos^2\phi d\phi\end{aligned}$$

The integral can be evaluated using a table of integrals or a computer program such as Mathematica, Maple or Maxima, to give:

$$\begin{aligned}\langle\delta_{z,i}^2\rangle &= \frac{l^2}{2}\int_0^\pi \sin\phi\cos^2\phi d\phi \\ &= \frac{l^2}{2}\left(-\frac{1}{3}\cos^3\phi\right)\Big|_0^\pi \\ &= \frac{l^2}{2}\left(\frac{1}{3}+\frac{1}{3}\right) = \frac{l^2}{3}\end{aligned}$$

[1] One could define the probability distribution function for ϕ as a constant, but this would give a different distribution of directions, in which those aligned more closely with the z-axis would be disproportionately favored.

We can now calculate the mean-square projection onto the z-axis of the end-to-end distances for a large number of n-step random walks:

$$\langle z_n^2 \rangle = n \langle \delta_{z,i}^2 \rangle = n \frac{l^2}{3}$$

Since the z-axis is not special (except for the definitions of ϕ and θ, which are arbitrary) the mean-square end-to-end projections onto the x- and $y-$ axis are also equal to $nl^2/3$, and the mean-square end-to-end distance is given by:

$$\begin{aligned} \langle r^2 \rangle &= \langle x^2 \rangle + \langle y^2 \rangle + \langle z^2 \rangle \\ &= n\frac{l^2}{3} + n\frac{l^2}{3} + n\frac{l^2}{3} \\ &= nl^2 \end{aligned}$$

Thus, we have exactly the same result as for one and two dimensions. In fact, the same result applies to random walks in any number of dimensions, though it may be hard to visualize the ones in more than three dimensions.

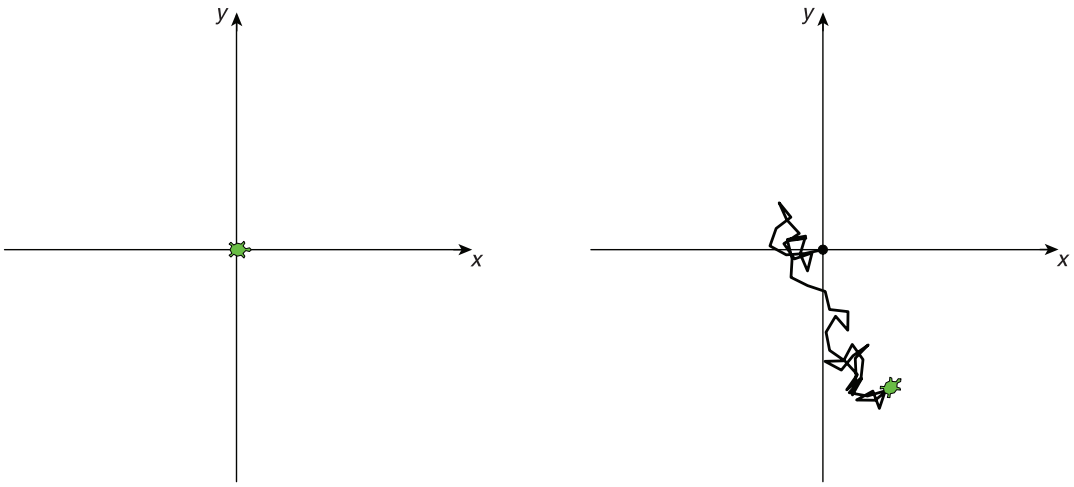
3.4 Computer Simulations of Random Walks

For many random processes, computer simulations can provide a useful complement to theoretical treatment, such as those derived in the previous sections. For random walks, simulations can provide snap shots of individual random walks and illustrate the distribution of properties, such as the end-point position, rather than just the averages that we have calculated so far. In addition, the process of writing the computer code for a simulation can help clarify one's thinking about the description of a random process.

I. Turtle graphics to illustrate two-dimensional random walks

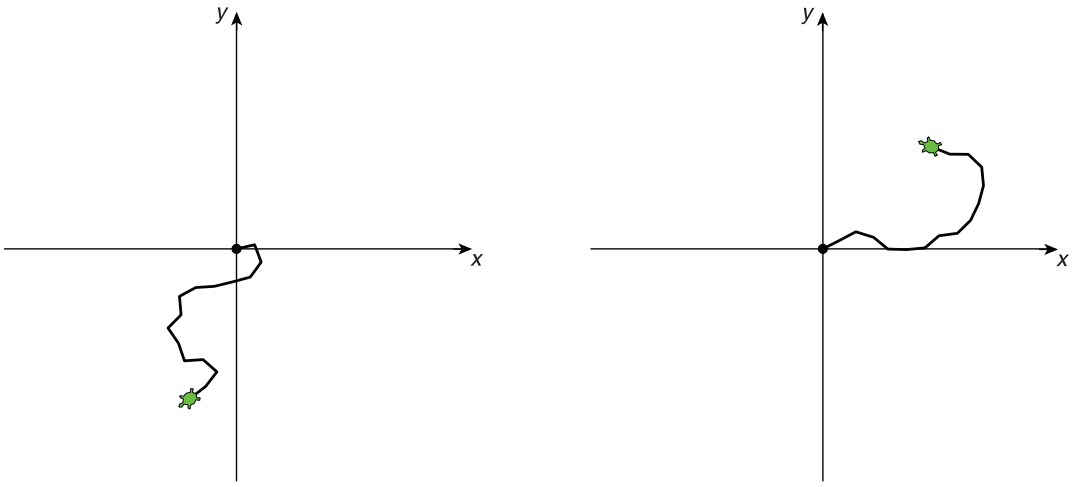
   Generating the high-quality computer graphics that we are now so familiar with takes a great deal of computer programming. But, there is a simple system that can be used for generating a graphical representation of two-dimensional random walks, called *turtle graphics*. Turtle graphics was introduced as a feature of a computer programming language, Logo, that was developed in the 1960s as a means for introducing children to computers and programming. Although there was much more to Logo than turtle graphics, that is probably the feature that it is best known for, and it has been adopted in other languages as well. The basic idea is that we imagine a turtle placed on a floor covered with a big piece of paper, and the turtle caries a pen. The turtle is given simple commands, such as "move forward by 10 units", or "turn right by 45°". These commands can be incorporated into programs where they are repeated numerous times, with variations, to generate a wide variety of patterns. Since a two-dimensional random walk consists of repeated steps and turns, turtle graphics is an ideal way to represent individual walks.

   The figures below show turtle-graphics representations of a turtle at its starting point, at the origin of the x-y coordinate system, (on the left) and after taking a taking a 200-step random walk (on the right).



The path of the turtle in this example illustrate an important general feature of random walks that is not readily apparent from the mathematical treatments of the previous two sections: The movement of the walker tends to be concentrated in small areas for a number of steps, followed by a series of steps in approximately the same direction, leading to a substantial excursion. The excursions are analogous to a series of coin tosses for which all, or nearly all, land heads. Although such series are relatively rare, they do occur on occasion, and have a significant effect when they do.

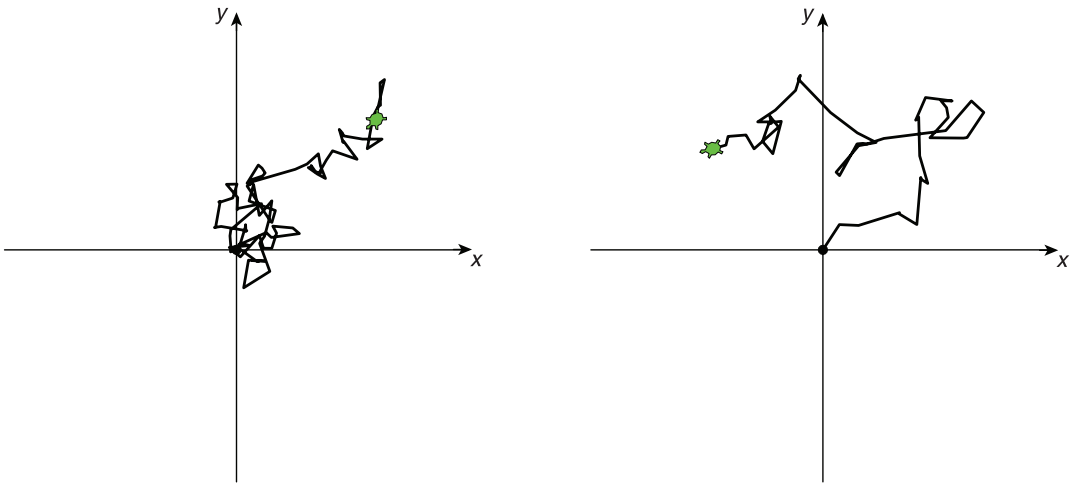
Another way in which simulations of this type are useful is that they let us explore the effects of changing the rules defining a random walk. For instance, the figures below represent random walks in which the turn angle at each step is constrained relative to the direction of the previous step. In the walk represented on the left, the turn from the previous was restricted to $\pm 90°$. In the right-hand figure, the turn was restricted to $\pm 46°$.



The obvious effects of restricting the turn angle in this way are to expand the random walk and reduce the number of times the path crosses itself. Note that these random walks consisted of only 20 steps each, yet the distance from the starting point is about

the same as for the 200-step random walk shown above, in which the turn angles were unrestricted. A random walk of this type is sometimes referred to as a *correlated random walk*.

We can also change the random walk by allowing the step length, as well as the turn angle, to vary randomly. The examples shown below were generated by sampling the step length from a Gaussian distribution centered at zero. For the diagram on the left, the standard deviation of the length distribution was 20 pixels, whereas the standard deviation was 30 pixels for the example on the right.

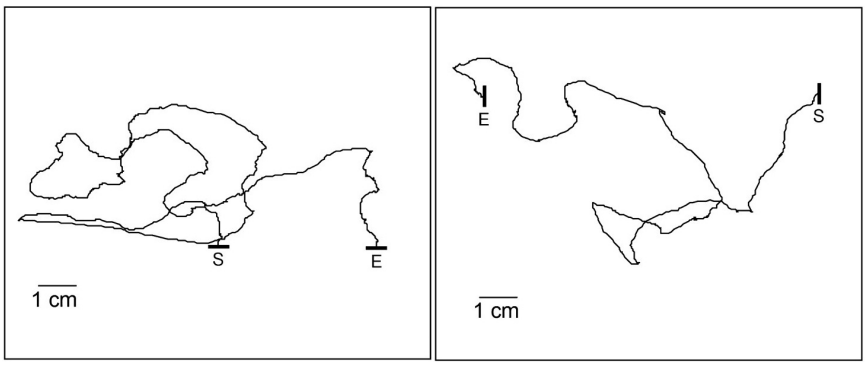


In both cases, short step lengths are the most probable, but longer ones are not uncommon. For the example on the left, with the narrower distribution, the random walk included 100 steps. To keep the walk to roughly the same extension, the number of steps was reduced to 50 for the walk shown on the right.

One of the important ways in which simulations of nearly any process can be used is to compare them to experimental observations. The extent to which the simulations matches the observed results can help support a theoretical model or indicate the ways in which the model might be improved. As an example the theory of random walks can be applied to analyzing the paths that animals follow when foraging for food.

The figures below represent the paths of individual ants, of two different species, as they foraged for food, as studied by Prof. Donald Feener and his colleagues at the University of Utah.[2]

[2]Pearce-Duvet, J. M. C., Elemens, C. P. H. and Feener, D. H. (2011) Walking the line: search behavior and foraging success in ant species. *Behavioral Ecology* 22, 501–509.



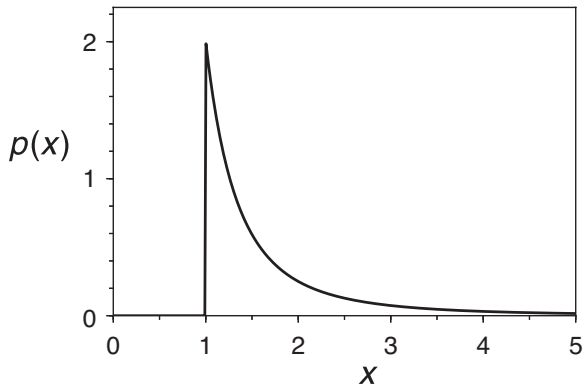
Brachymyrmex depilis (25 s) *Dorymyrmex insanus* (21 s)

The paths of the ants show some distinct similarities with those generated in the simulations shown above. In particular, there are frequent turns with a wide range of angles, and relatively straight segments of varying lengths. Of course, an ant has to take a large number of tiny steps to cover any significant distance, but it is reasonable to define random walk steps that correspond to the relatively straight segments in the path.

Particularly for *Brachymyrmex deplis*, the range of step lengths is extremely wide. A random walk model that has been used to describe walks with occasional steps that are very long is called a *Lévy flight*. This type of walk is characterized by a *long-tailed distribution* of step sizes, meaning that long steps are favored much more than by a Gaussian distribution. One such distribution is called the Pareto function, which has the general form:

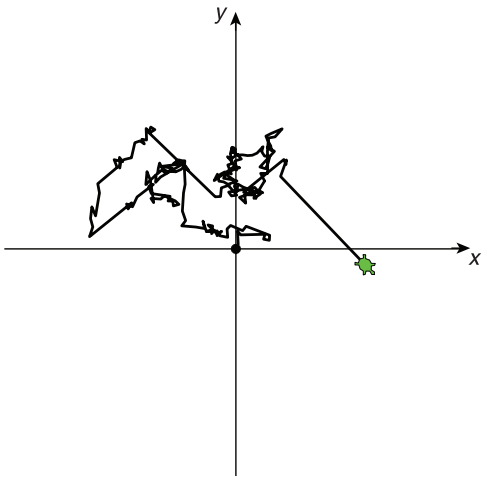
$$p(x) = \begin{cases} \frac{\alpha x_{\mathrm{m}}^{\alpha}}{x^{\alpha+1}} & \text{if } x \geq x_{\mathrm{m}} \\ 0 & \text{if } x < x_{\mathrm{m}} \end{cases}$$

where x_{m} is the minimum value for which the probability is greater than zero. The parameter α determines the rate at which the probability falls as x increases and lies between 0 and 2. An example of a Pareto distribution, with $x_{\mathrm{m}} = 1$ and $\alpha = 2$ is plotted below.



As written above, the Pareto function is a normalized probability distribution function. An interesting property of this function is that for $\alpha \leq 2$, its variance is infinite. (More properly, the integral representing the variance increases without bound as x increases.)

The figure below shows a simulated Lévy flight based on the Pareto function with the step lengths determined by a Pareto function with $x_{\mathrm{m}} = 10$ and $\alpha = 2$.

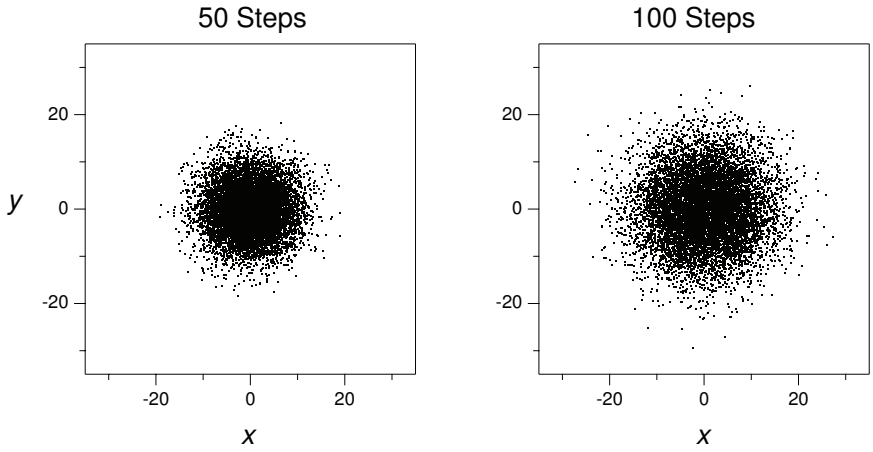


Compared to the random walks shown earlier, this one is characterized by a few very long steps, separated by much more localized random steps. In this respect, it seems to be a better model for the behavior of the ants shown above, and there is some evidence that this type of random walk is appropriate model for the foraging behavior of many species.

It should be noted that each of the turtle graphics representations shown above is just a single random walk and was chosen without complete objectivity. In particular the examples were chosen to highlight particular features, and for the fact that they didn't exceed the arbitrary boundaries of the axes. None the less, the features highlighted are quite real and can be found when large samples are examined.

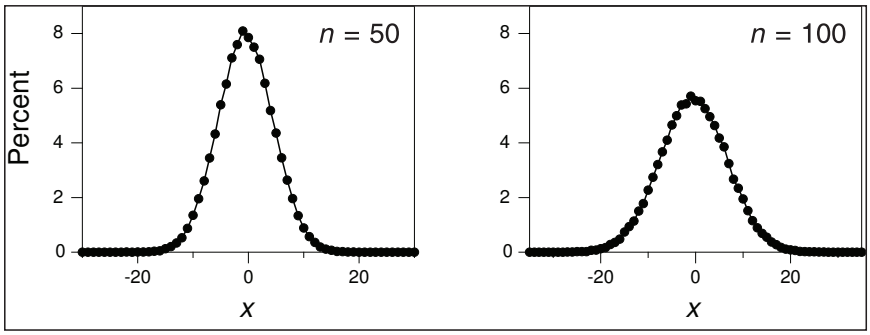
II. Simulating large samples of random walks

Another way in which simulations can be used is to generate large samples of random walks, from which general statistical insights can be gained. The figure below shows the endpoints of 10,000 two-dimensional random walks of 50 and 100 steps, on the left and right respectively. The steps in these random walks have a length of 1, in arbitrary units.



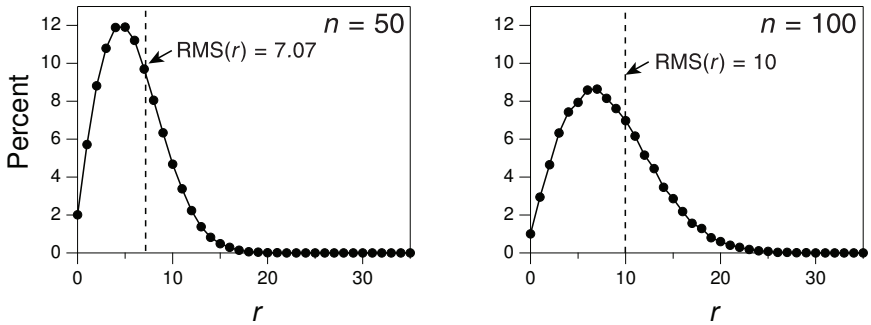
Note that the maximum projection along the x- or y-axes is 50 or 100, for the left and right panels, respectively. However, even among 10,000 random walks, distances greater than 20 are rare in either case.

The graphs below show the relative probabilities of the x-projections of the endpoints lying within intervals 1 step-length wide.

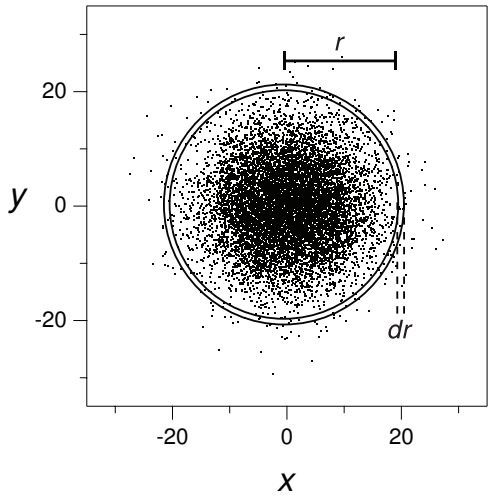


To produce these relatively smooth curves, 100,000 simulated random walks were generated. As expected, the distributions are bell shaped and centered at $x = 0$. As the number of steps increases, the breadth of the distribution increases.

The next set of graphs, below, show the distribution of distances between the starting and end points.



It may seem surprising that the peaks of these distributions are not at $r = 0$, since the highest density of endpoints in near the starting point. To understand this apparent paradox, it is important to keep in mind the meaning of a probability distribution function. If the probability distribution function is $p(r)$, then the probability that the distance lies with in a small interval of r-values is the product $p(r)dr$, as represented in the figure below:



As shown in the figure, the probability distribution function, $p(r)$, represents the probability that the endpoint of the random walk lies within an annulus dr thick at a distance r from the starting point. This probability depends on both the density of points at distance r *and* the area of the annulus. This area is calculated as:

$$A = 2\pi r dr$$

To help visualize this relationship, imagine a thin metal ring, with radius r and thickness dr. If you were to cut this ring and flatten it out, the cross-sectional area (viewed along the thin edge) would be the length, $2\pi r$, times the thickness dr. Thus, the area of the annulus increases as r increases, while the density of endpoints decreases with r. The product of the area and the density is small when either term is small, and reaches a maximum at an intermediate value of r, as shown in the distribution function.