Physical Principles in Biology
Biology 3550
Spring 2024

Lecture 10:

The One-dimensional Random Walk

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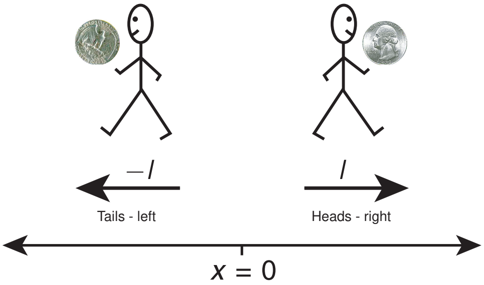
A Random Walk in One Dimension



1. Start at position $x = 0$.
2. Flip coin.
   - Heads, take step of length l to the right.
   - Tails, take step of length l to the left.
3. Repeat 2 another $(n - 1)$ times.

- Final position is x_n.

- What factors will determine the distribution of x_n for a large number of random walks?
  - The step length, l.
  - The number of steps, n
  - The probability of a step to the right or left.

Calculate The Average Final Position (The Expected Value of x_n)

- For a single random walk, the final position will be:

$$x_n = \sum_{i=1}^{n} \delta_i$$

  where i is the step number, and δ_i is the displacement along the x-axis for step i and is either $+l$ or $-l$, with probabilities p_+ and p_-.

- For each step, δ_i is a random variable, with an expected value, $E(\delta_i)$:

$$\begin{aligned} E(\delta_i) &= lp_+ + (-lp_-) = lp_+ - lp_- \\ &= l(p_+ - p_-) \\ &= l\big(p_+ - (1 - p_+)\big) \\ &= l(2p_+ - 1) \end{aligned}$$

Calculating The Expected Value of x_n

- An important theorem: If x and y are two **independent** random variables, then the expected value of the sum is calculated as:

$$E(x+y) = E(x) + E(y)$$

- Since:

$$x_n = \sum_{i=1}^{n} \delta_i$$

The expected value of x_n is calculated as:

$$E\left(x_n\right) = \sum_{i=1}^{n} E(\delta_i) = nE(\delta_i)$$

$$= nl\left(2p_+ - 1\right)$$

Clicker Question #1

If the random-walk step size is 0.5 m, and the probability of a forward step, p_+, is 0.3, what is the expected value of x_n for a 50-step random walk?

A) -10 m

B) -5 m

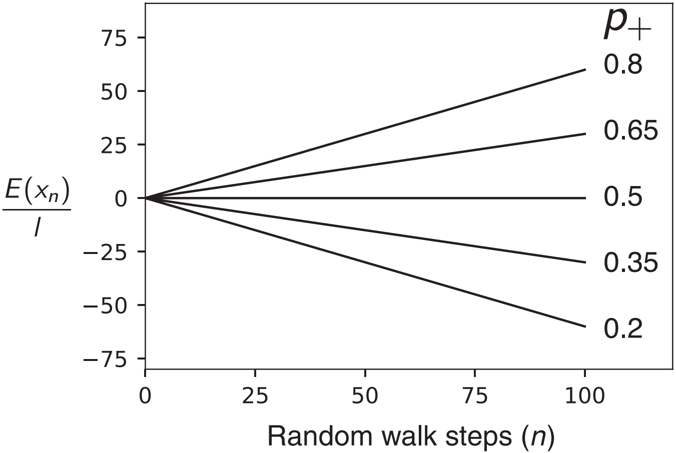
C) 0 m

D) 5 m

E) 10 m

$$\begin{aligned} E(x_n) &= nl(2p_+ - 1) \\ &= 50 \times 0.5\,\mathrm{m}(2p_+ - 1) = 50 \times 0.5\,\mathrm{m}(2 \cdot 0.3 - 1) \\ &= 50 \times 0.5\,\mathrm{m} \times -0.4 = -10\,\mathrm{m} \end{aligned}$$

Expected Value of x_n for a One-dimensional Random Walk



Some Different Kinds of Average

For N random walks of n steps each:

- The mean:

$$\langle x_n \rangle = \frac{1}{N} \sum_{j=1}^{N} x_{n,j}, \text{ for large } N$$

  Angle brackets, $\langle \ \rangle$, indicate average over a large sample.
  $x_{n,j}$ is the final position of the j^{th} walk.

- The mean-square average:

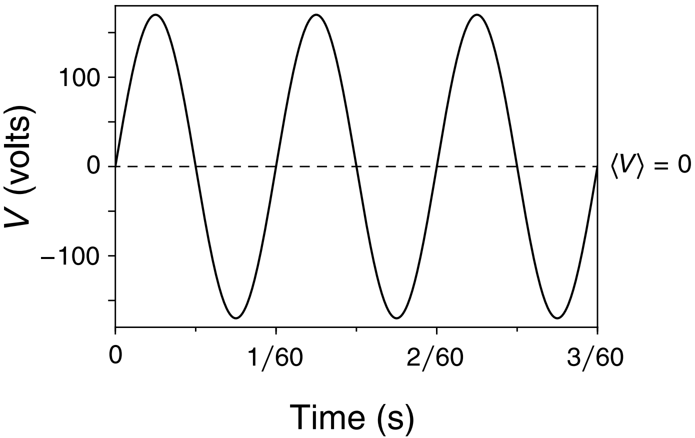
$$\langle x_n^2 \rangle = \frac{1}{N} \sum_{j=1}^{N} x_{n,j}^2$$

- The root-mean-square (RMS) average:

$$\text{RMS}(x_n) = \sqrt{\langle x_n^2 \rangle} = \sqrt{\frac{1}{N} \sum_{j=1}^{N} x_{n,j}^2}$$

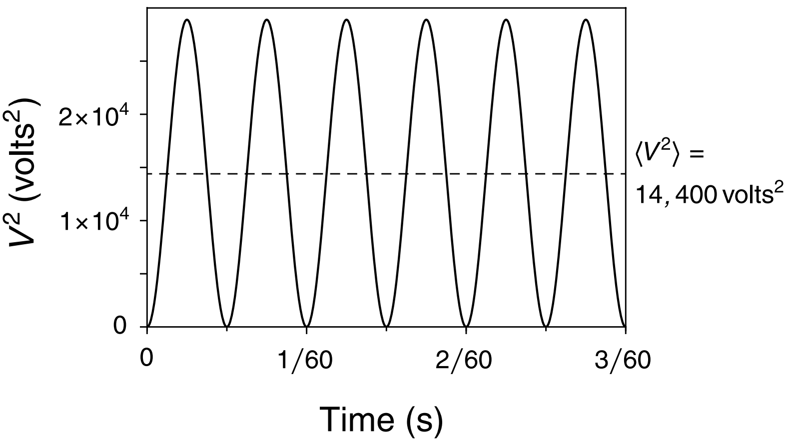
An Application of Mean-square and Root-mean-square Averages: Household Power (US)

Voltage versus time



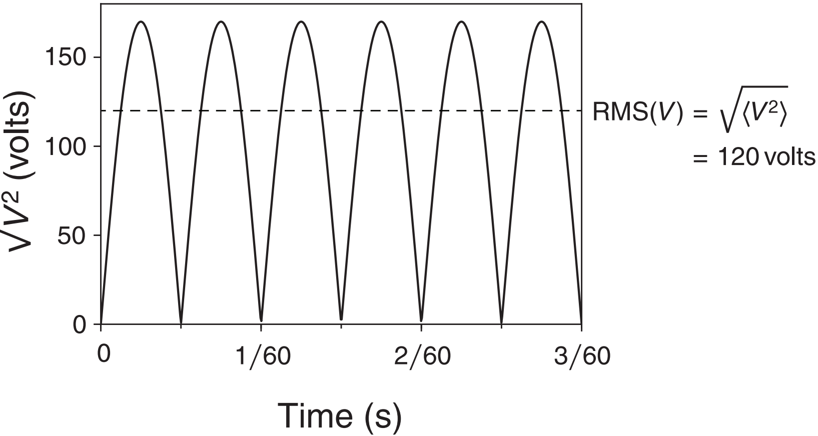
An Application of Mean-square and Root-mean-square Averages: Household Power (US)

Voltage squared versus time



An Application of Mean-square and Root-mean-square Averages: Household Power (US)

$\sqrt{V^2}$ versus time



Clicker Question #2

For the numbers: $-4, 2, -3, 1, 5,$
Calculate the root-mean-square average

A) ~ 0.2

B) ~ 1.5

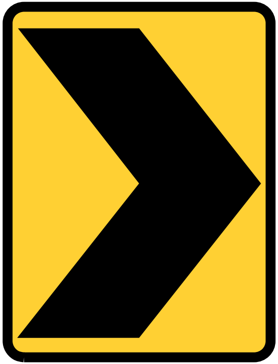
C) ~ 2.9

D) ~ 3.3

E) ~ 4.8

$$\text{RMS} = \sqrt{\frac{-4^2 + 2^2 + -3^2 + 1^2 + 5^2}{5}} = \sqrt{\frac{16 + 4 + 9 + 1 + 25}{5}} = \sqrt{\frac{55}{5}} = \sqrt{11}$$

Warning!



Direction Change

Back to the one-dimensional random walk

Calculating the Mean-Square Displacement for a 1-d Random Walk

- For a single random walk, the final position will be:

  $$x_n = \sum_{i=1}^{n} \delta_i$$

  where i is the step number, and δ_i is either $-l$ or $+l$.

- Here, we will assume that positive and negative steps are equally probable.

- We can also express x_n in terms of the position after the next-to-last step, x_{n-1}:

  $$x_n = x_{n-1} + \delta_n$$

  Something tricky is coming up!

Calculating The Mean-Square Displacement

- If we do a large number, N, of random walks, the mean-square displacement, $\langle x \rangle$, will be:

$$\langle x_n^2 \rangle = \frac{1}{N} \sum_{j=1}^{N} x_{n,j}^2 = \frac{1}{N} \sum_{j=1}^{N} \left(\sum_{i=1}^{n} \delta_{j,i} \right)^2$$

  where j is the random walk number, and $\delta_{j,i}$ is the displacement for step i of walk j.

- We can also write the mean-square average as:

$$\langle x_n^2 \rangle = \frac{1}{N} \sum_{j=1}^{N} \left(x_{(n-1),j} + \delta_{j,n} \right)^2$$

$$= \frac{1}{N} \sum_{j=1}^{N} \left(x_{(n-1),j}^2 + 2x_{(n-1),j}\delta_{j,n} + \delta_{j,n}^2 \right)$$

Calculating The Mean-Square Displacement

- Following from the previous slide:

$$\langle x_n^2 \rangle = \frac{1}{N}\sum_{j=1}^{N}\left(x_{(n-1),j}^2 + 2x_{(n-1),j}\delta_{j,n} + \delta_{j,n}^2\right)$$

$$= \frac{1}{N}\sum_{j=1}^{N} x_{(n-1),j}^2 + \frac{1}{N}\sum_{j=1}^{N}(2x_{(n-1),j}\delta_{j,n} + \frac{1}{N}\sum_{j=1}^{N}\delta_{j,n}^2$$

$$= \langle x_{n-1}^2 \rangle + \langle 2x_{n-1}\delta_n \rangle + \langle \delta_n^2 \rangle$$

- For a large number of unbiased walks:
  - For each walk, δ_n is either $+l$ or $-l$, with equal probability, and is independent of the position before the last step, x_{n-1}.
  - The average of $2x_{n-1}\delta_n$ over N walks, $\langle 2x_{n-1}\delta_n \rangle$, is 0.

Calculating The Mean-Square Displacement

- From the previous slide:

$$\begin{aligned}\langle x_n^2\rangle &= \langle x_{n-1}^2\rangle + \langle 2x_{n-1}\delta_n\rangle + \langle \delta_n^2\rangle \\ &= \langle x_{n-1}^2\rangle + \langle \delta_n^2\rangle\end{aligned}$$

- Following the same logic:

$$\langle x_{n-1}^2\rangle = \langle x_{n-2}^2\rangle + \langle \delta_{n-1}^2\rangle$$

- and

$$\begin{aligned}\langle x_n^2\rangle &= \langle x_{n-2}^2\rangle + \langle \delta_{n-1}^2\rangle + \langle \delta_n^2\rangle \\ &= \langle x_{n-2}^2\rangle + 2\langle \delta^2\rangle\end{aligned}$$

Calculating The Mean-Square Displacement

- Continuing in the same way:

$$\langle x_n^2 \rangle = \langle x_{n-2}^2 \rangle + 2\langle \delta^2 \rangle$$

$$= \langle x_{n-3}^2 \rangle + \langle \delta_{n-2}^2 \rangle + 2\langle \delta^2 \rangle$$

$$= \langle x_{n-3}^2 \rangle + 3\langle \delta^2 \rangle$$

$$\langle x_n^2 \rangle = \langle x_{n-4}^2 \rangle + 4\langle \delta^2 \rangle$$

- and so on, until we have:

$$\langle x_n^2 \rangle = \langle x_1^2 \rangle + (n-1)\langle \delta^2 \rangle$$

$$= \langle x_0^2 \rangle + n\langle \delta^2 \rangle$$

$$= n\langle \delta^2 \rangle$$

- A derivation based on recursion!

The Mean-square Displacement for One Step: $\langle \delta^2 \rangle$

- The average of δ^2 over a large number of steps is equal to the expected value of δ^2:

$$
\begin{aligned}
\langle \delta^2 \rangle = E(\delta^2) &= p_+(+l)^2 + p_-(-l)^2 \\
&= p_+ l^2 + p_- l^2 \\
&= l^2 \left(p_+ + p_-\right) \\
&= l^2
\end{aligned}
$$

- The mean-square displacement for the random walk:

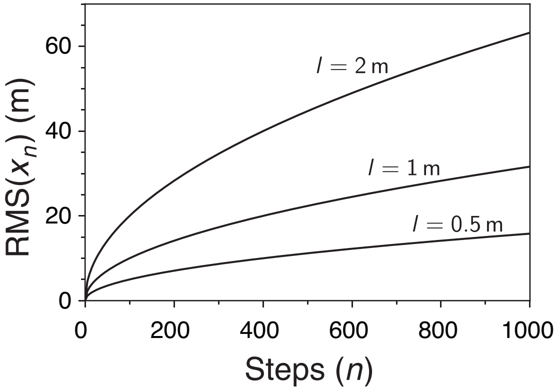
$$\langle x_n^2 \rangle = n l^2$$

- The root-mean-square displacement for the random walk:

$$\mathrm{RMS}(x_n) = \sqrt{\langle x_n^2 \rangle} = \sqrt{n}\, l$$

The Root-mean-square Displacement for a One-dimensional Random Walk

$$\mathrm{RMS}(x_n) = \sqrt{\langle x_n^2 \rangle} = \sqrt{n}\,l$$



- This is THE most important thing to remember about random walks!

Clicker Question #3

For a one-dimensional random walk of 100 steps, of length 0.25 m, what is the total distance traveled, as measured by a pedometer (or a FitBit)?

A) 2.5 m

B) 5 m

C) 8 m

D) 25 m

E) 50 m

$$\begin{aligned} \text{Total distance} &= nl \\ &= 100 \times 0.25\,\text{m} = 25\,\text{m} \end{aligned}$$

Clicker Question #4

For a one-dimensional random walk of 100 steps, of length 0.25 m, what is the expected RMS distance between the starting and ending points?

A) 2.5 m

B) 5 m

C) 8 m

D) 25 m

E) 50 m

$$\begin{aligned}\mathrm{RMS}(x_n) &= \sqrt{n}\,l \\ &= \sqrt{100} \times 0.25\,\mathrm{m} = 10 \times 0.25\,\mathrm{m} = 2.5\,\mathrm{m}\end{aligned}$$

Clicker Question #5

For a one-dimensional random walk of 10 steps, of length 2.5 m, what is the total distance traveled, as measured by a pedometer (or a FitBit)?

A) 2.5 m

B) 5 m

C) 7.9 m

D) 25 m

E) 50 m

$$\text{Total distance} = nl$$
$$= 10 \times 2.5\,\text{m} = 25\,\text{m}$$

Clicker Question #6

For a one-dimensional random walk of 10 steps, of length 2.5 m, what is the expected RMS distance between the starting and ending points?

A) 2.5 m

B) 5 m

C) 8 m

D) 25 m

E) 50 m

$$\begin{aligned} \mathrm{RMS}(x_n) &= \sqrt{n}\,l \\ &= \sqrt{10} \times 2.5\,\mathrm{m} \approx 3.162 \times 2.5\,\mathrm{m} \approx 7.9\,\mathrm{m} \end{aligned}$$