Physical Principles in Biology
Biology 3550
Spring 2024

Lecture 12:

Simulating Random Processes and

Two-dimensional Random Walks

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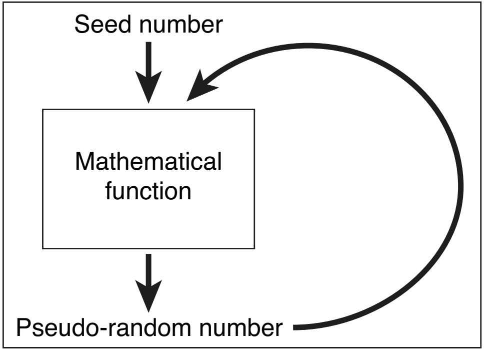
Simulating Random Processes with a Computer

- Computers aren't supposed to do things at random!
- But, we often ask them to!
- Pseudo-random number generators



- Function has to be carefully designed so that numbers "look random".

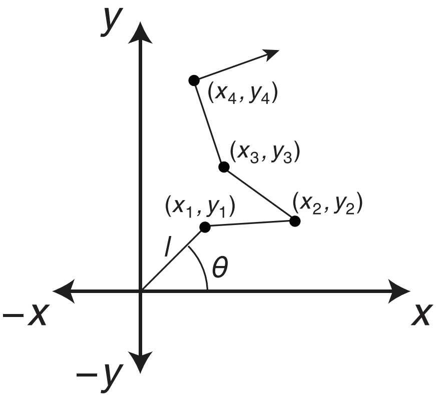
How Do We Decide if Numbers "Look Random"?

- After generating lots of numbers, they should approximate a defined distribution; for instance evenly distributed numbers from 0 to 1.
- Shouldn't be able to predict one number from a previous one, without knowing the algorithm.
- A sign of trouble: Numbers start repeating.
  - Eventually this will happen with any pseudo-random number generator.
  - Repeat period should be very large.
    (greater than the number of numbers to be used)

Where Does the Seed Number Come From?

- A user-specified number, to generate a predictable set of pseudo-random numbers. Useful for simulations.
- The computer's clock. Very common method.
- A truly random physical process:
  - Radioactive decay.
    `https://www.fourmilab.ch/hotbits/`
  - A lava lamp!
    `https://en.wikipedia.org/wiki/Lavarand`
  - Electronic or thermal noise.
    `https://en.wikipedia.org/wiki/Hardware_random_number_generator`
    Now incorporated in many computer CPUs and USB dongles.
- Good random numbers are are becoming more important every day!

A Random Walk in Two Dimensions

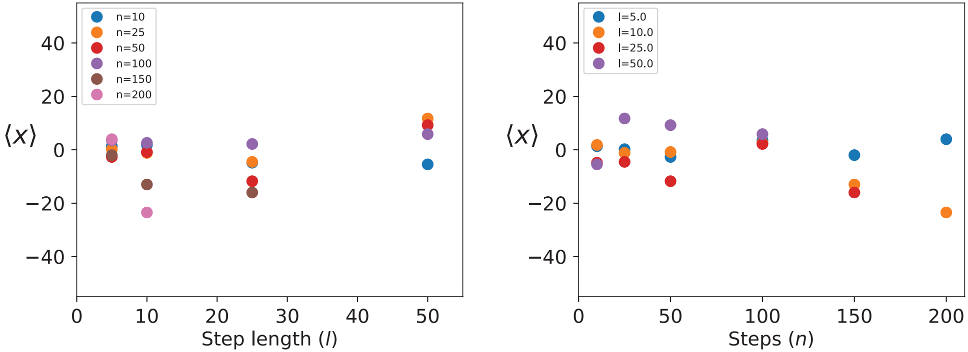


1. Start at (x, y) coordinates (0,0).
2. Choose a random direction, defined by the angle θ from the x-axis.
3. Move distance l in the chosen direction.
4. Repeat another $(n - 1)$ times.

Important Parameters for a Two-Dimensional Random Walk

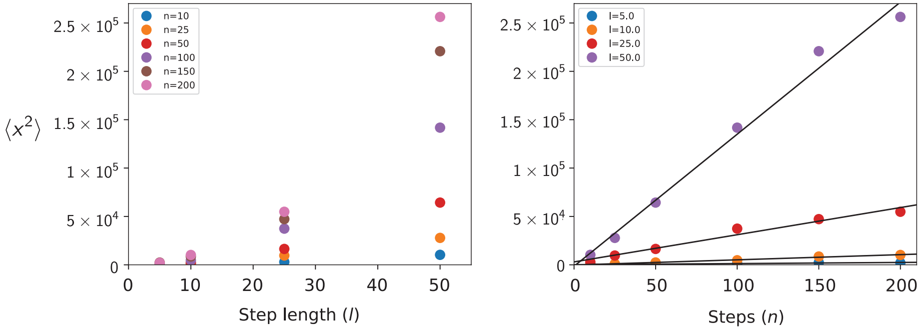
- l: The step length. Fixed for our simulations.
- θ_i: Turn angle for step i. Uniformly distributed from 0 to 2π radians for our simulations.
- n: Number of steps in a single random walk.
- N: Total number of random walks used for averaging. 100 in our simulations.
- x_n and y_n: Final x and y-coordinates of random walk.
- r_n: Distance between start and end of random walk.
- $\langle x \rangle$, $\langle y \rangle$ and $\langle r \rangle$: Mean values of x_n, y_n and r_n, over a large number of walks.
- $\langle x^2 \rangle$, $\langle y^2 \rangle$ and $\langle r^2 \rangle$: Mean-square averages of x_n, y_n and r_n.
- $\mathrm{RMS}(x)$, $\mathrm{RMS}(y)$ and $\mathrm{RMS}(r)$: Root-mean-square averages of x_n, y_n and r_n.

$\langle x \rangle$ versus l and n



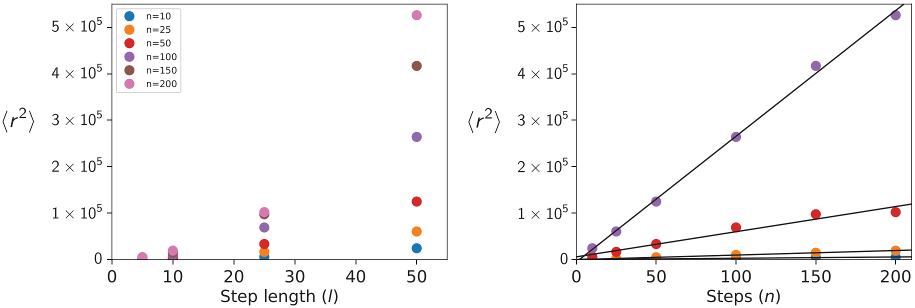
- Values cluster around 0.
- $\langle y \rangle$ looks just like $\langle x \rangle$.

$\langle x^2 \rangle$ versus l and n



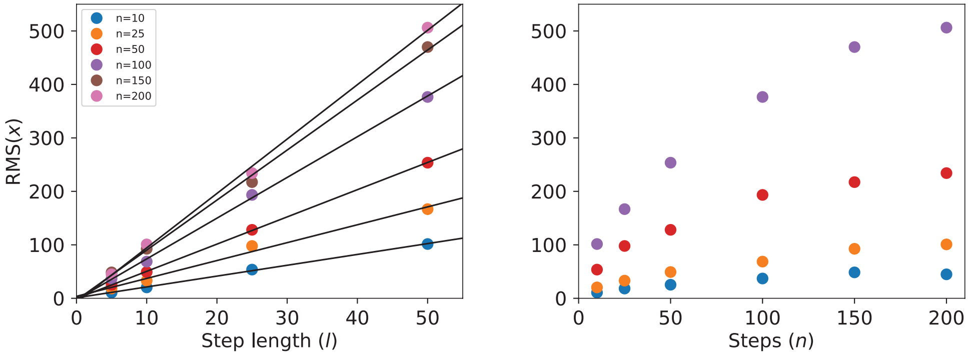
- $\langle x^2 \rangle$ is proportional to n.
- Increase in $\langle x^2 \rangle$ with l is not linear.
- $\langle y^2 \rangle$ looks just like $\langle x^2 \rangle$.

$\langle r^2 \rangle$ versus l and n



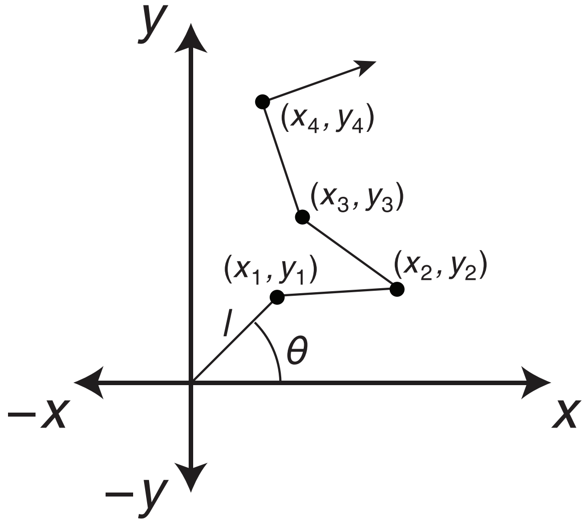
- $\langle r^2 \rangle$ is proportional to n.
- Increase in $\langle r^2 \rangle$ with l is not linear.

RMS(x) versus l and n

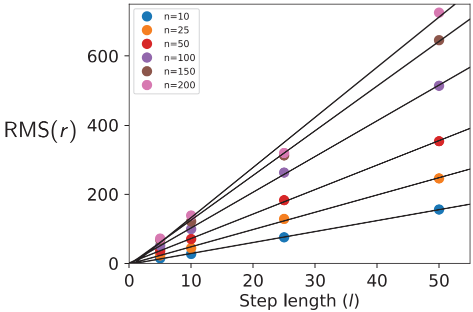


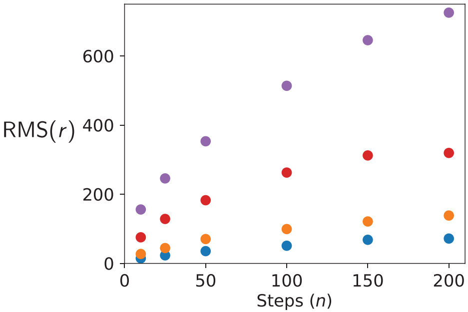
- RMS(x) is proportional to l
- Increase in RMS(x) with n is not linear.
- RMS(y) looks just like RMS(x)

A Random Walk in Two Dimensions



RMS(r) versus l and n





- RMS(r) is proportional to l
- The increase in RMS(r) with n is not linear.
- Looks a lot like x_n in one-dimensional random walk!

A Summary

Observed proportionalities:

$$\langle x^2 \rangle \propto n$$
$$\langle y^2 \rangle \propto n$$
$$\langle r^2 \rangle \propto n$$

$$\mathrm{RMS}(x) \propto l$$
$$\mathrm{RMS}(y) \propto l$$
$$\mathrm{RMS}(r) \propto l$$

Inferred proportionalities:

$$\mathrm{RMS}(x) \propto \sqrt{n}$$
$$\mathrm{RMS}(y) \propto \sqrt{n}$$
$$\mathrm{RMS}(r) \propto \sqrt{n}$$

$$\langle x^2 \rangle \propto l^2$$
$$\langle y^2 \rangle \propto l^2$$
$$\langle r^2 \rangle \propto l^2$$

$$\langle x^2 \rangle \propto nl^2 \qquad \langle y^2 \rangle \propto nl^2 \qquad \langle r^2 \rangle \propto nl^2$$
$$\mathrm{RMS}(x) \propto \sqrt{n}l \qquad \mathrm{RMS}(y) \propto \sqrt{n}l \qquad \mathrm{RMS}(r) \propto \sqrt{n}l$$

- What are the constants of proportionality? (slopes)

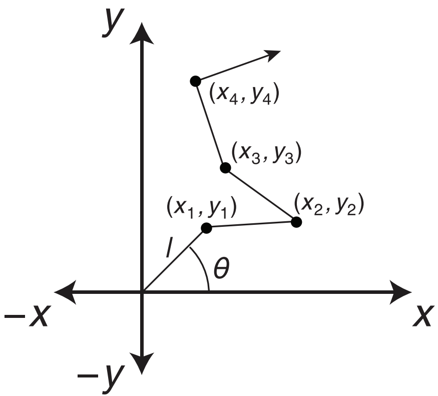
Theory Says:

For a two-dimensional random walk:

$$\langle x^2 \rangle = \frac{n}{2} l^2 \qquad \langle y^2 \rangle = \frac{n}{2} l^2 \qquad \langle r^2 \rangle = n l^2$$

$$\mathrm{RMS}(x) = \sqrt{\frac{n}{2}}\, l \qquad \mathrm{RMS}(y) = \sqrt{\frac{n}{2}}\, l \qquad \mathrm{RMS}(r) = \sqrt{n}\, l$$

Clicker Question #1



For an unbiased 2-dimensional rw of 75 steps of length 0.5 m, what is the expected value of the final x-coordinate, $\langle x \rangle$

A) 0 m

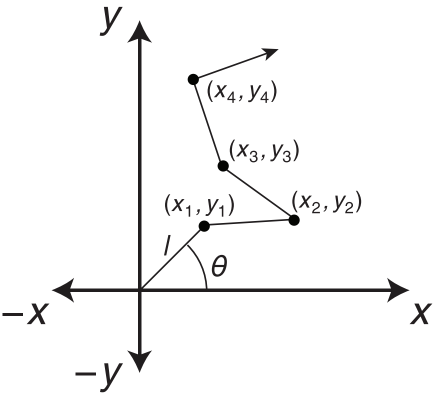
B) 3.1 m

C) 4.3 m

D) 9.4 m

E) 18.8 m

Clicker Question #2



For an unbiased 2-dimensional rw of 75 steps of length 0.5 m, what is the expected value of the square of the final x-coordinate, $\langle x^2 \rangle$?

A) $0\,\mathrm{m}^2$

B) $3.1\,\mathrm{m}^2$

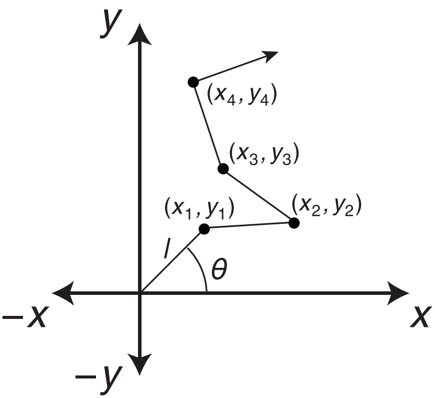
C) $4.3\,\mathrm{m}^2$

D) $9.4\,\mathrm{m}^2$

E) $19.4\,\mathrm{m}^2$

$$\langle x_n^2 \rangle = \frac{n}{2} l^2 = \frac{75}{2}(0.5\,\mathrm{m})^2 = 9.4\,\mathrm{m}^2$$

Clicker Question #3



For an unbiased 2-dimensional rw of 75 steps of length 0.5 m, what is the expected root-mean-square end-to-end distance, RMS(r)?

A) 0 m

B) 3.1 m

C) 4.3 m

D) 9.4 m

E) 18.8 m

$$\text{RMS}(r^2) = l\sqrt{n} = 0.5\,\text{m} \times \sqrt{75} = 4.3\,\text{m}$$

Clicker Question #4

A turtle walks into a bar and, after a few hours, walks out and starts on a random walk! The turtle walks straight for a period, turns in a random direction, walks straight again, and repeats.

After watching the random walk for several hours, an observer concludes that:

- The turtle walks at a remarkably steady pace of 0.5 m/min.
- The time interval between turns is also very consistent.
- The RMS overall distance traveled by the turtle in 30 min, measured along a straight line, (RMS(r)) is 4 m.

For how long does the turtle walk between turns?

A) ≈ 1 min

B) ≈ 2 min

C) ≈ 5 min

D) ≈ 10 min

E) ≈ 20 min