Physical Principles in Biology
Biology 3550
Spring 2024

Lecture 13:

Variations on the Two-dimensional Random Walk and Continuous Probability Distribution Functions

Wednesday, 7 February 2024

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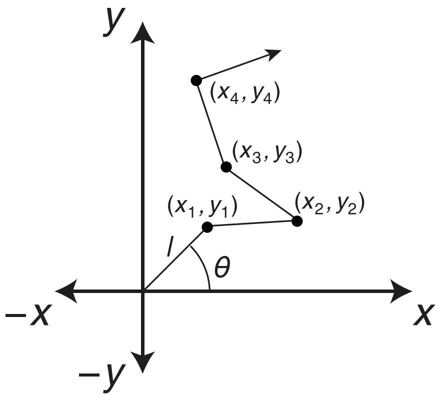
Announcements

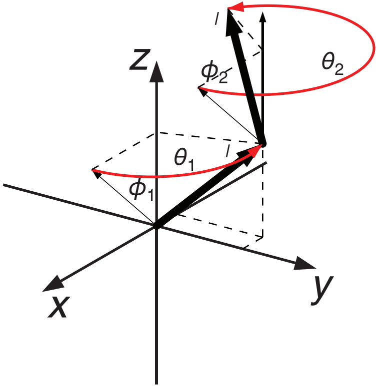
- Problem Set 2:
  - Due 11:59 PM, Monday, 12 February.
  - Download problems from Canvas.
  - Upload work to Gradescope.
  - Show your work!
  - Please don't scrunch things up!
- Quiz 2:
  - Friday, 9 February
  - 25 min, second half of class.
- Review Session
  - 5:15 PM, Thursday, 8 February
  - HEB 2002
  - Come with questions!

The Unbiased Random Walk in Two Dimensions



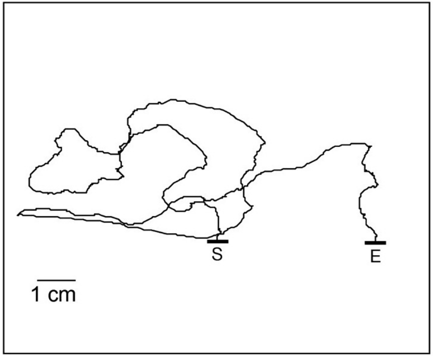
- $\langle x_n^2 \rangle = \langle y_n^2 \rangle = nl^2/2$
- $\langle r_n^2 \rangle = nl^2$

Description of a Three-dimensional Random Walk

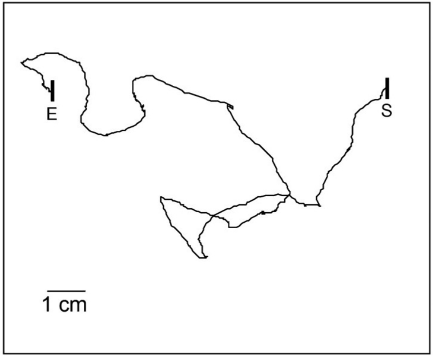


- Each step is defined by a tilt from the local z-axis (ϕ_i) and a rotation around the z-axis (θ_i).
- The end of each step lies on a sphere of radius l.
- $\langle x_n^2 \rangle = \langle y_n^2 \rangle = \langle z_n^2 \rangle = nl^2/3$
- $\langle r^2 \rangle = nl^2$, and $\mathrm{RMS}(r) = \sqrt{n}l$, just like in one and two dimensions.

Ants on a Walk for Food



Brachymyrmex depilis
(25 s)



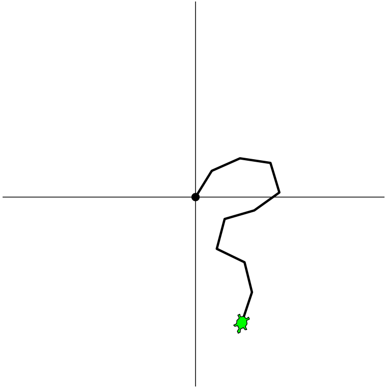
Dorymyrmex insanus
(21 s)

- Do either look like a random walk?

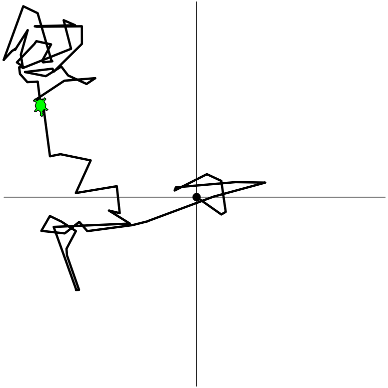
Pearce-Duvet, J. M. C., Elemens, C. P. H. & Feener, D. H. (2011). Walking the line: search behavior and foraging success in ant species. *Behavioral Ecology*, 22, 501–509.
http://dx.doi.org/10.1093/beheco/arr001

Simple Variations on the Two-dimensional Random Walk

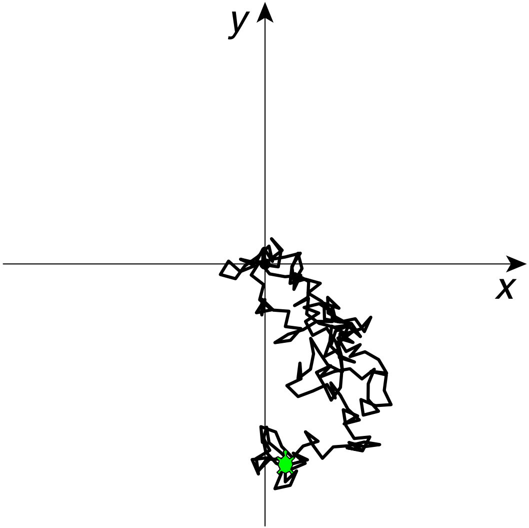
- Constrain change in direction.



- Introduce variation in step length.

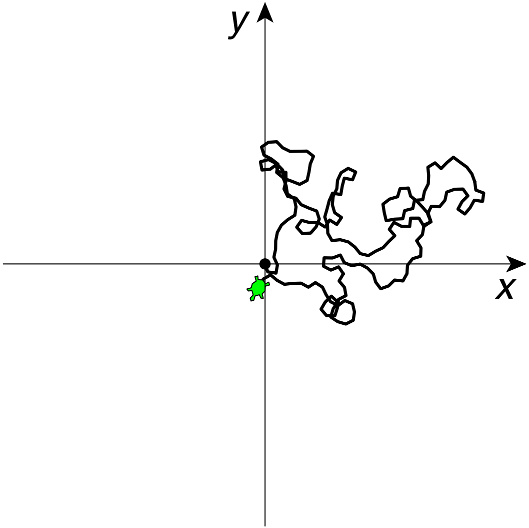


A ‘Plain’ Random Walk



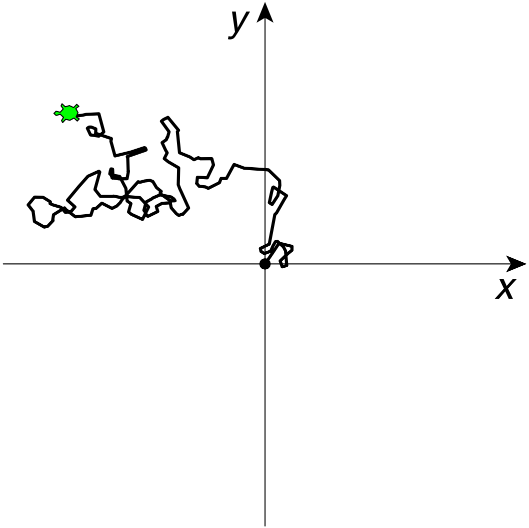
- Step length = 20
- No. steps = 200

A “Correlated” Random Walk



- Turn angle restricted to -90° to 90°
- Step length = 8
- No. steps = 200

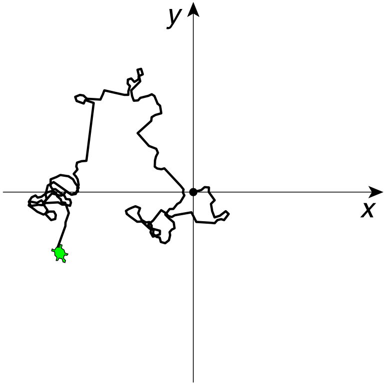
A Random Walk With a Distribution of Step Lengths



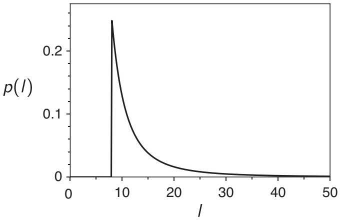
- Turn angle restricted to -90° to 90°
- Half-Gaussian (bell curve) distribution of step lengths
- No. steps = 200

A “Lévy Flight”

A random walk with a “heavy-tailed” distribution of step lengths



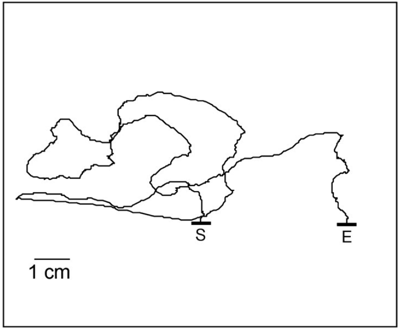
- Turn angle restricted to -90° to 90°
- Pareto distribution of step lengths, for Vilfredo Pareto (1842–1923)



- No. steps = 200

Clicker Question #1

What does the ant walk most resemble?



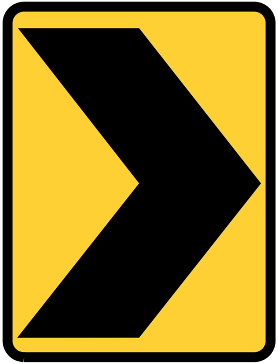
Brachymyrmex depilis
(25 s)

A) A plain random walk

B) A correlated random walk

C) A Lévy Flight

Warning!



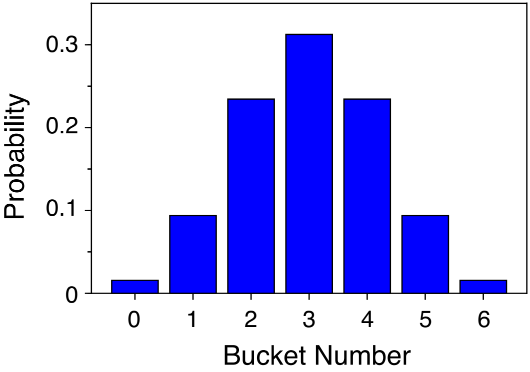
Direction Change

Continuous Probability Distribution Functions

Discrete Probability Distribution Functions

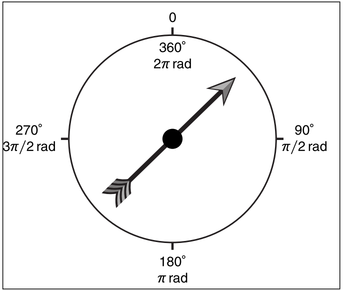
- For random processes with discrete outcomes.
- Variables take on discrete values.
- The probability distribution functions can be viewed as tables or bar graphs

Bucket No.	Probability
0	1/64
1	6/64
2	15/64
3	20/64
4	15/64
5	6/64
6	1/64



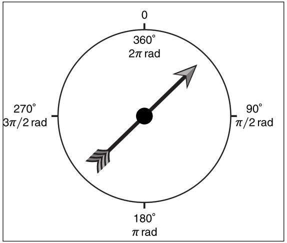
Introducing Continuous Probability Distribution Functions

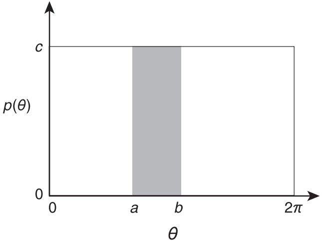
- A spinner to choose directions for the 2-dimensional random walk



- We could divide up the circle into a finite number of sectors.
  - Two sectors: Like flipping a coin
  - Six sectors: Like throwing a die
  - Lots of other possibilities
- OR, we can treat the result as a continuous variable from 0 to 2π rad

A Continuous Probability Distribution Function for the Spinner



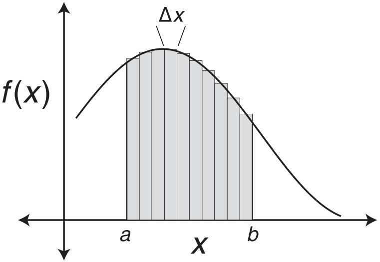


- θ is a continuous variable, with values from 0 to 2π.
- $p(\theta)$ is a function of θ, with a constant value, c, for all values of θ.
- Interpretation of $p(\theta)$: The integral

$$\int_a^b p(\theta)d\theta$$

is the probability that the spinner lands between the values a and b.

A Quick Refresher of Integrals (as "area under the curve")



- To approximate the area between the x-axis and the function $f(x)$, between $x = a$ and $x = b$:
  - Divide up the range $a \leq x \leq b$ into n segments $\Delta x = (b - a)/n$ wide.
  - Draw n rectangles Δx wide and $f(x_i)$ high.
  - Sum the areas of the rectangles

$$\text{area} \approx \sum_{i=1}^{n} f(x_i)\Delta x$$

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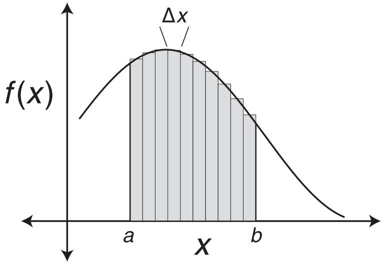

- Improve approximation by making Δx smaller (and n larger).
- If the function is "well behaved", Δx can be made infinitesimally small.
- The definite integral, from a to b with respect to x, is defined as:

$$\int_a^b f(x)dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\Delta x$$

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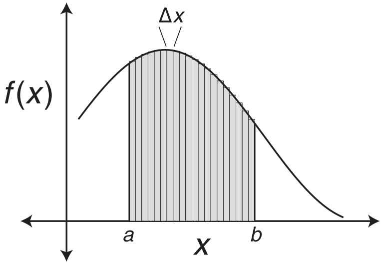

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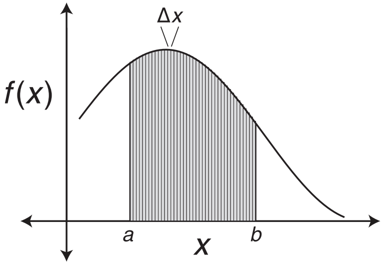

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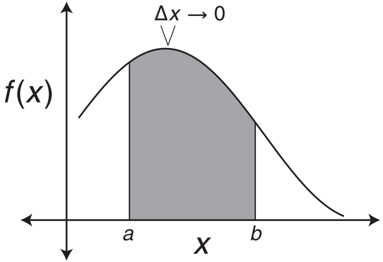

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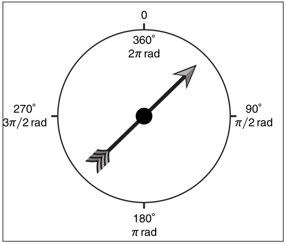
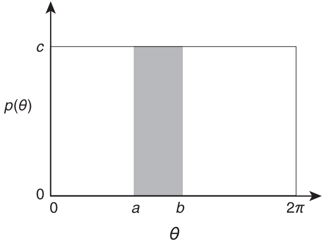
Back to the Continuous Probability Distribution Function (PDF) for the Spinner

- $p(\theta)$ is a function of θ, with a constant value, c, for all values of θ.



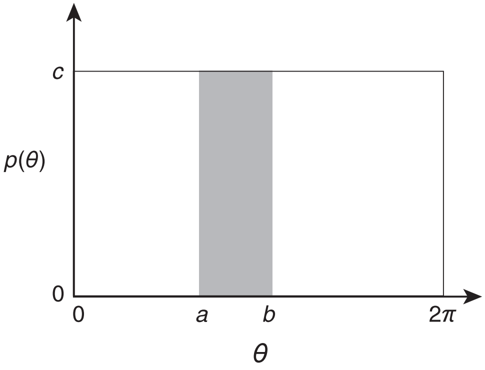


- The integral

$$\int_a^b p(\theta)d\theta = \int_a^b c\, d\theta$$

is the probability that the spinner lands between the values a and b.

An Important Constraint on a Continuous PDF



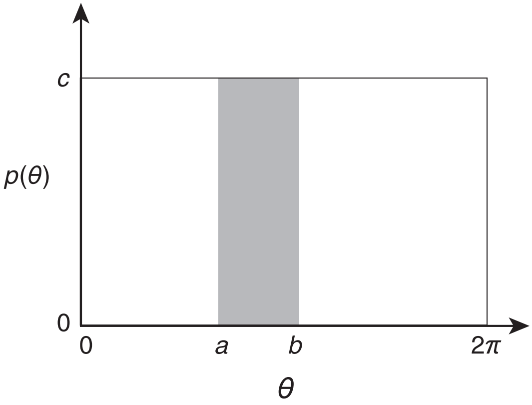
- To be make sense, the integral over all possible values must equal 1:

$$\int_0^{2\pi} p(\theta) d\theta = 1$$

- Equivalent to the requirement for a discrete PDF that the sum of all probabilities be equal to 1.
- For the spinner pdf, the constant, c, is chosen to normalize the PDF.

$$p(\theta) = c$$

Clicker Question #2



What value of c should be used to normalize the spinner PDF, so that:

$$\int_0^{2\pi} p(\theta) d\theta = 1$$

A) 0

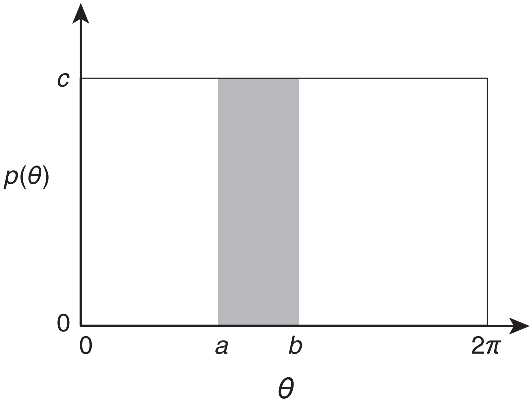
B) $\frac{1}{2\pi}$

C) 1

D) π

E) 2π

Choosing the Constant



What value of c should be used to normalize the spinner PDF?, so that:

$$\int_0^{2\pi} p(\theta) d\theta = 1$$

$$\int_0^{2\pi} c d\theta = 1$$

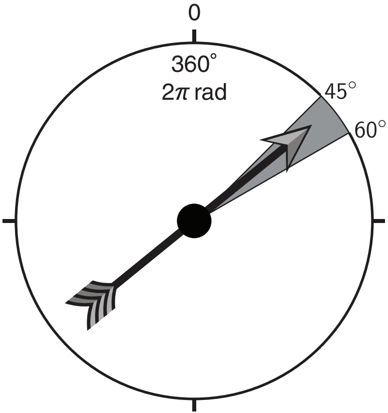
$$c\theta\Big|_0^{2\pi} = c \cdot 2\pi - c \cdot 0 = 1$$

$$c 2\pi = 1$$

$$c = \frac{1}{2\pi}$$

Clicker Question #3

What is the probability that the spinner will lie between 45°and 60°?



A) ≈ 0.02

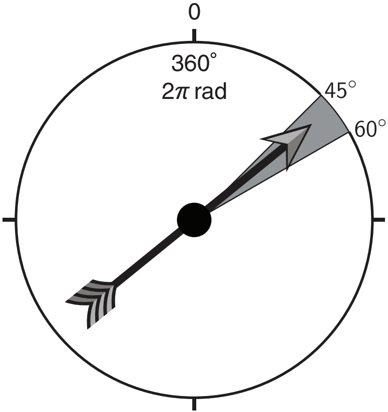
B) ≈ 0.04

C) ≈ 0.06

D) ≈ 0.08

E) ≈ 0.1

The probability that the spinner will lie between 45°and 60°



- $45^\circ = \pi/4 \, \text{rad}$
- $60^\circ = \pi/3 \, \text{rad}$

$$p = \int_{\pi/4}^{\pi/3} p(\theta) d\theta = \int_{\pi/4}^{\pi/3} \frac{1}{2\pi} d\theta$$

$$= \frac{1}{2\pi}\theta \Big|_{\pi/4}^{\pi/3} = \frac{1}{2\pi}\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

$$= \frac{1}{2\pi}\left(\frac{4\pi}{12} - \frac{3\pi}{12}\right) = \frac{1}{24} \approx 0.04$$

The Expected Value (Mean) for a Continuous PDF

- For a discrete random variable, x, with discrete PDF, $p(x)$, the expected value is:

$$E(x) = \mu = \sum_{i=1}^{n} p(x_i) x_i$$

- For a continuous random variable, x, with range $x_1 \leq x \leq x_2$ and continuous PDF, $p(x)$, the expected value is:

$$E(x) = \mu = \int_{x_1}^{x_2} p(x) x dx$$

- For the spinner variable, θ:

$$E(\theta) = \mu = \int_{0}^{2\pi} p(\theta) \theta d\theta = \int_{0}^{2\pi} \frac{1}{2\pi} \theta d\theta = \frac{1}{4\pi} \theta^2 d\theta \Big|_{0}^{2\pi} = \pi$$