Physical Principles in Biology
Biology 3550
Spring 2024

Lecture 7:

Plinko Probabilities

Wednesday, 24 January 2024

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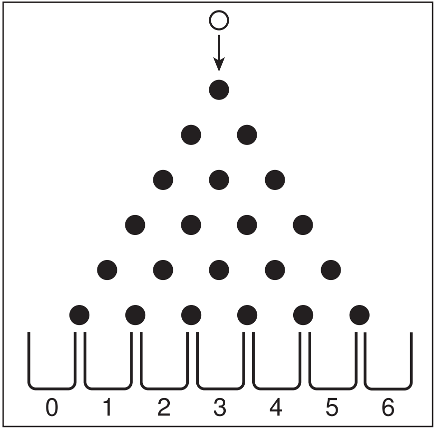
Announcements

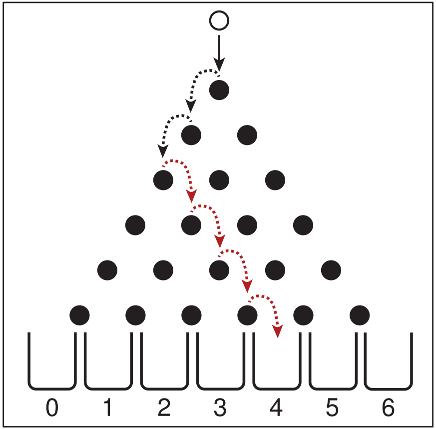
- No office hours on Thursday, 25 January:
- Review Session:
  - Thursday, 25 January: 5:15 PM
  - HEB 2002
  - Come with questions!
- Quiz 1:
  - Friday, 26 January
  - 25 min, second half of class.
  - Bring a calculator!

Plinko Probabilities: A Six-row Plinko



- The question: What is the probability that a ball will fall in each of the buckets.
- For N plinko rows, there will be $N + 1$ buckets for balls to land in.
- The buckets are numbered from 0 to N.
- Define the sample set as all of the possible paths.
- Define events as all the paths to a given bucket.

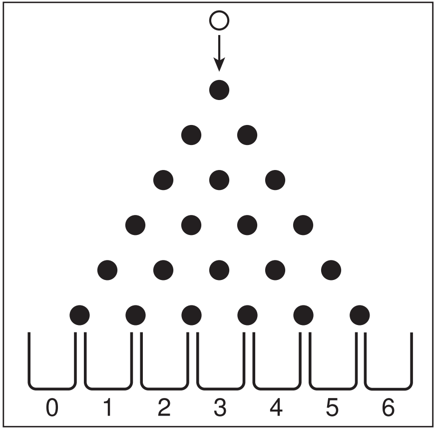
From Counting Paths to buckets 0, 1, 2 , 4, 5 and 6



Bucket No.	Paths
0	1
1	6
2	15
3	
4	15
5	6
6	1

Clicker Question #1

How many paths are there to bucket 3?



A) 20

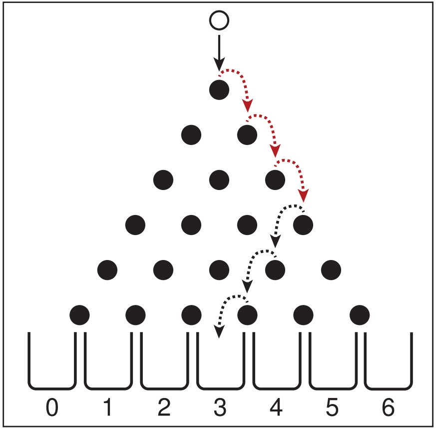
B) 25

C) 30

D) 35

E) 40

How Many Paths to Bucket 3



Bucket No.	Paths
0	1
1	6
2	15
3	20
4	15
5	6
6	1

- The total number of paths is 64.
- Counting the paths to bucket 3 looks hard!

Another Way to Count the Paths to Bucket 2

- 2 turns to the right and 4 turns to the left.
- There are 6 rows where one turn could be placed.
- There are 5 rows where a second turn could be placed.
  $6 \times 5 = 30$
- BUT, this assumes that the turns can be placed in either order!
- The first turn has to come before the second, so each possible path has been counted twice.
- The correct count: $6 \times 5 \times \frac{1}{2} = 15$
- A general strategy: Count all of the possible places where each right turn could be placed, allowing all possible orders, and then correct for over counting.

Counting the Paths to Bucket 3

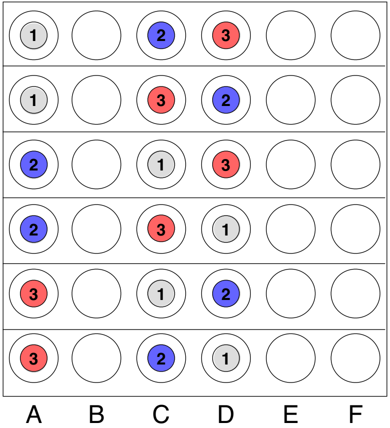
- 3 turns to the right and 3 to the left.
- Ignoring the order of placement:
  - 6 rows where a first right turn can be placed.
  - 5 rows where a second right turn can be placed.
  - 4 rows where a third right turn can be placed.
  - $6 \times 5 \times 4 = 120$
- But, this assumes turns can be placed in any order!
- How do we determine how many times the paths have been over counted?

A Related Problem: Placing Beans in Cups

- Suppose that we have 3 beans, each with the number 1, 2 or 3 printed on it.
- How many different ways are there to place the beans in 6 cups? (with no more than one bean per cup)
  - 6 cups where the first bean can be placed.
  - 5 cups where the second bean can be placed.
  - 4 cups where the third bean can be placed.
  - $6 \times 5 \times 4 = 120$
- These are all different, because the beans are distinguishable and can be placed in any order.
- It is also the number of over-counted paths to bucket 3 in the plinko.
- How many of the arrangements have the beans in a specific order, say "1, 2, 3"?

3 Labeled Beans in 3 Cups

- How many ways are there to put 3 labeled beans in 3 **specific** cups?



- $3 \times 2 \times 1 = 6$
- Only one of these has the order 1-2-3.

Three Labeled Beans in Six Cups

- If all orders are counted:

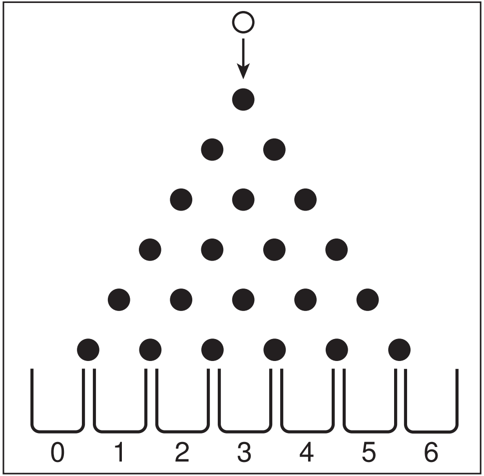
  Number of ways to place beans is: $6 \times 5 \times 4 = 120$

- If only the placements with the order 1-2-3 are counted:

  Number of ways to place beans is: $(6 \times 5 \times 4) \div 6 = 20$

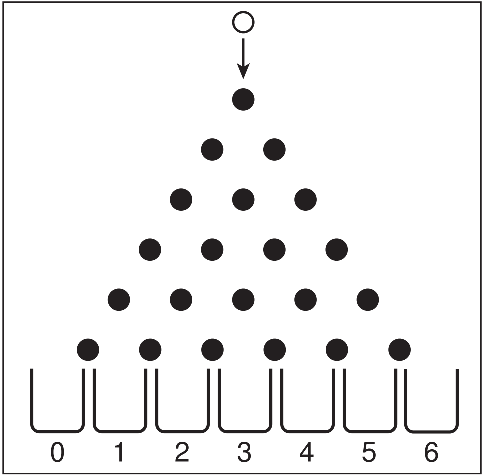
- Also the number of ways to place 3 turns to the right
  in the 6-row plinko!

The Full Path Count for the Six-row Plinko



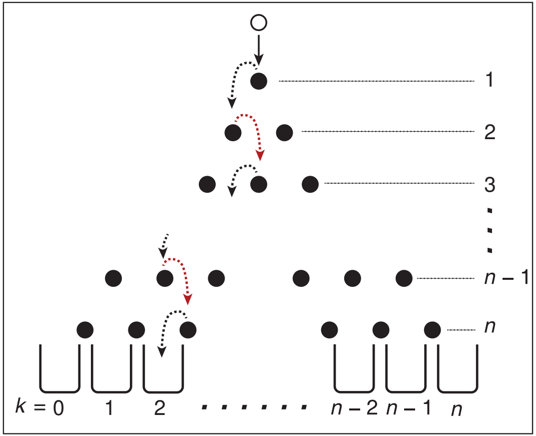
Bucket No.	Paths
0	1
1	6
2	15
3	20
4	15
5	6
6	1

Probabilities for the Six-row Plinko



Bucket No.	Paths	Probability
0	1	$1/64 \approx 0.016$
1	6	$6/64 \approx 0.094$
2	15	$15/64 \approx 0.234$
3	20	$20/64 \approx 0.312$
4	15	$15/64 \approx 0.234$
5	6	$6/64 \approx 0.094$
6	1	$1/64 \approx 0.016$

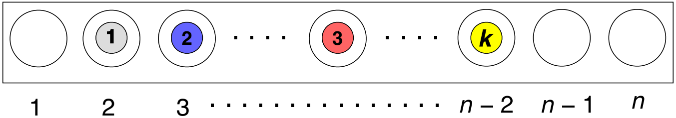
An n-row Plinko



- k = bucket number.
- To reach bucket k, ball must make k turns to the right and $n - k$ turns to the left.

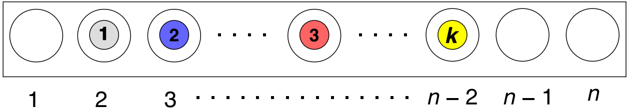
Beans and Cups

- For an n-row plinko, the number of paths to bucket k is the number of ways to place k labeled beans in n cups **in a single order**.



- To calculate this number:
  1. Calculate the number of ways to place k labeled beans in n cups, **in any order**.
  2. Calculate the number of ways to place k labeled beans in k cups, in any order.
  3. Divide result of 1 by result of 2.

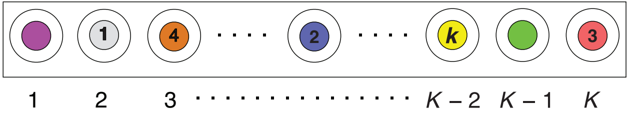
The Number of Ways to Place k Labeled Beans in n Cups, in Any Order



- The first bean can go in any of n cups.
- The second bean can go in any of the $n - 1$ cups that are left.
- The third bean can go in any of the $n - 2$ cups that are left.
- When it is time to find a place for the k^{th} bean:
  - $k - 1$ of the cups have beans in them.
  - The number of cups without beans is $n - (k - 1) = n - k + 1$
- The total number of ways to place the k labeled beans is:

$$n(n-1)(n-2)\cdots(n-k+1)$$

The Number of Ways to Place k Labeled Beans in k Cups, in Any Order



- The first bean can go in any of k cups.
- The second bean can go in any of the $k-1$ cups that are left.
- The third bean can go in any of the $k-2$ cups that are left.
- The k^{th} bean only has one place to go!
- The total number of ways to place the k labeled beans is:

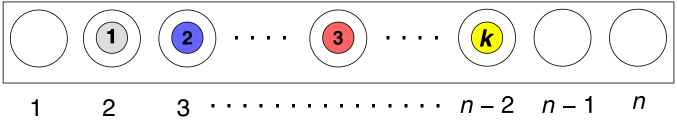
  $k(k-1)(k-2)\cdots 2\cdot 1$

The Factorial Function

$$k! = \begin{cases} k(k-1)(k-2)\cdots 2 \cdot 1 & \text{if } k > 0; \\ 1 & \text{if } k = 0 \end{cases}$$

- $k!$ is only defined for integers greater than or equal to zero.
- Why is $0! = 1$?
- $k!$ is the number of distinct ways of placing k distinguishable objects in k positions.

The Number of Ways to Place k Labeled Beans in n Cups, in Any Order



- Previously showed that the total number of ways to place the k labeled beans is:

$$n(n-1)(n-2)\cdots(n-k+1)$$

- Can rewrite this as:

$$\frac{n(n-1)\cdots(n-k+1)(n-k)(n-k-1)\cdots 2\cdot 1}{(n-k)(n-k-1)\cdots 2\cdot 1} = \frac{n!}{(n-k)!}$$

Back to the Plinko

- For an n-row plinko, the number of paths to bucket k is the number of ways to place k labeled beans in n cups **in a single order**.
- The number of ways to put k labeled beans in n cups, in any order, is:

$$n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

- The number of ways to place k labeled beans in k cups is in any order:

$$\frac{k!}{(k-k)!} = k!$$

- The number of ways to place k labeled beans in n cups **in a single order** is:

$$\frac{n!}{k!(n-k)!}$$

- The universal plinko formula!

Test the Recipe on the Six-Row Plinko

- Paths to bucket 0: $\dfrac{n!}{k!(n-k)!} = \dfrac{6!}{0!6!} = \dfrac{720}{720 \cdot 1} = 1$
- Paths to bucket 6: $\dfrac{n!}{k!(n-k)!} = \dfrac{6!}{6!0!} = \dfrac{720}{720 \cdot 1} = 1$
- Paths to bucket 1: $\dfrac{n!}{k!(n-k)!} = \dfrac{6!}{1!5!} = \dfrac{720}{120} = 6$
- Paths to bucket 5: $\dfrac{n!}{k!(n-k)!} = \dfrac{6!}{5!1!} = \dfrac{720}{120 \cdot 1} = 6$

Test the Recipe on the Six-Row Plinko (contd.)

- Paths to bucket 2: $$\frac{n!}{k!(n-k)!} = \frac{6!}{2!4!} = \frac{720}{2 \cdot 24} = 15$$
- Paths to bucket 4: $$\frac{n!}{k!(n-k)!} = \frac{6!}{4!2!} = \frac{720}{24 \cdot 2} = 15$$
- Paths to bucket 3: $$\frac{n!}{k!(n-k)!} = \frac{6!}{3!3!} = \frac{720}{6 \cdot 6} = 20$$
- It works!
  Now we can do any size plinko!

Clicker Question #2

For a 7-row plinko, with 8 buckets labeled 0 to 7,
what is the probability of a ball landing in bucket 1?

A) ~ 0.01

B) ~ 0.05

C) ~ 0.1

D) ~ 0.15

E) ~ 0.2

$$p(1) = \frac{n!}{k!(n-k)!} \cdot 2^{-n} = \frac{7!}{1!(7-1)!} \cdot 2^{-7} = \frac{7!}{6!} \cdot 2^{-7} = 7 \cdot 2^{-7}$$

n choose k

- The expression we have derived applies to much more than plinkos!
- The expression is often written as:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

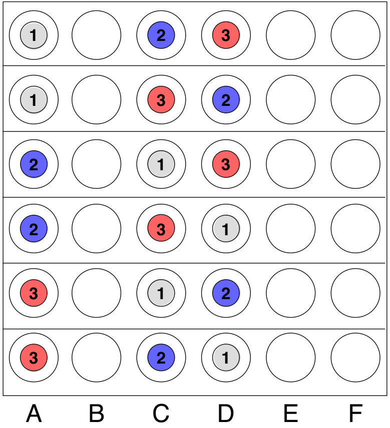
  and spoken as "n choose k"

- From n objects, choose k of them (each only once) and either:
  - The objects are labeled, and only a single order is allowed

    Or
  - The objects are unlabeled.

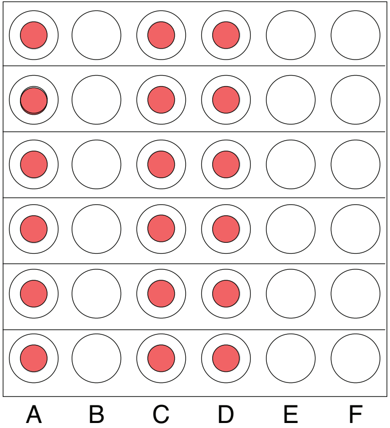
    But, not if
  - The objects are labeled and all orders are allowed.

3 Labeled Beans in 3 Cups



- $3 \times 2 \times 1 = 6$
- Only one of these has the order 1-2-3.
- What if we remove the labels?

3 Unlabeled Beans in 3 Cups



- All of these arrangements are the same!
- To get the number of arrangements of *k* unlabeled beans in *n* cups:
  - Count the ways to arrange *k* labeled beans in *n* cups, in any order.
  - Divide by the ways to arrange *k* labeled beans in *k* cups, in any order.

Binomial Coefficients

- The series of numbers generated by

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

  for a specific value of n and increasing values of $k \leq n$ are called "binomial coefficients."

- The binomial coefficients arise in algebra in the expansion of a sum of two terms (a binomial):

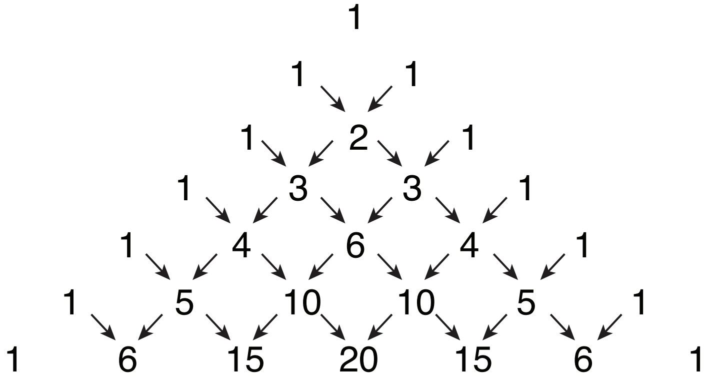
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$
$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$
$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

Pascal's Triangle

	Row
1	0
1 1	1
1 2 1	2
1 3 3 1	3
1 4 6 4 1	4
1 5 10 10 5 1	5
1 6 15 20 15 6 1	6 ,

- Blaise Pascal, French mathematician, 1623–1662
- Triangle was known long before Pascal's time; in India, Persia and China, among other places.
- But Pascal wrote a book about it.

Pascal's Triangle



- Start with 1s on left and right sides.
- Calculate other elements by adding two values above.