Lecture 8:
Plinko Probabilities, Part II
Binomial Coefficients and the Binomial Distribution Function

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### Probabilities for the Six-row Plinko

<table>
<thead>
<tr>
<th>Bucket No.</th>
<th>Paths</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>$1/64 \approx 0.016$</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>$6/64 \approx 0.094$</td>
</tr>
<tr>
<td>2</td>
<td>15</td>
<td>$15/64 \approx 0.234$</td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td>$20/64 \approx 0.312$</td>
</tr>
<tr>
<td>4</td>
<td>15</td>
<td>$15/64 \approx 0.234$</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
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<td>6</td>
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</tr>
</tbody>
</table>
An $n$-row Plinko

- $k =$ bucket number.

- To reach bucket $k$, ball must make $k$ turns to the right and $n - k$ turns to the left.
For an \( n \)-row plinko, the number of paths to bucket \( k \) is the number of ways to place \( k \) labeled beans in \( n \) cups in a single order.

To calculate this number:

1. Calculate the number of ways to place \( k \) labeled beans in \( n \) cups, in any order.
2. Calculate the number of ways to place \( k \) labeled beans in \( k \) cups, in any order.
3. Divide result of 1 by result of 2.
The Number of Ways to Place \( k \) Labeled Beans in \( n \) Cups, in a Single Order

- The number of ways to place \( k \) labeled beans in \( n \) cups in any order:

\[
n(n-1)(n-2)\cdots(n-k+1) = \\
\frac{n(n-1)\cdots(n-k+1)(n-k)(n-k-1)\cdots2\cdot1}{(n-k)(n-k-1)\cdots2\cdot1} = \frac{n!}{(n-k)!}
\]

- The number of ways to place \( k \) labeled beans in \( k \) cups in any order (\( k = n \)):

\[
\frac{k!}{(k-k)!} = k!
\]

- The number of ways to place \( k \) labeled beans in \( n \) cups in a single order is:

\[
\frac{n!}{(n-k)!} \cdot \frac{1}{k!} = \frac{n!}{k!(n-k)!}
\]
For a 5-row plinko, with 6 buckets labeled 0 to 5, how many paths are there to bucket 3?

A) 2
B) 4
C) 6
D) 8
E) 10

$$\frac{5!}{3!(5 - 3)!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(3 \cdot 2 \cdot 1)(2 \cdot 1)} = \frac{120}{12} = 10$$
The expression we have derived applies to much more than plinkos!

The expression is often written as:

\[ \binom{n}{k} = \frac{n!}{(n-k)!k!} \]

and spoken as “n choose k”

From n objects, choose k of them (each only once) and either

- Only a single order is allowed (e.g., turns in the plinko)
  Or
- The order doesn’t matter (e.g., unlabeled beans).
  But, not if
- The objects are labeled and all orders are allowed.
Binomial Coefficients

The series of numbers generated by

\[ \binom{n}{k} = \frac{n!}{(n-k)!k!} \]

for a specific value of \( n \) and increasing values of \( k \leq n \) are called “binomial coefficients.”

The binomial coefficients arise in algebra in the expansion of a sum of two terms:

\[
\begin{align*}
(a + b)^2 &= a^2 + 2ab + b^2 \\
(a + b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\
(a + b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\
(a + b)^5 &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 \\
(a + b)^6 &= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6
\end{align*}
\]
Pascal’s Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
Row
0
1
2
3
4
5
6
. . . . . . . . . . . . . . . . . . .
. . . . . . . . . . . . . . . .
. . . . . . . . . . . . .
. . . . . . . . . .
. . . . . . .
. . . .
. .

Blaise Pascal, French mathematician, 1623–1662

Triangle was known long before Pascal’s time, but Pascal wrote a book about it.
Pascal’s Triangle

- Start with 1s on left and right sides.
- Calculate other elements by adding two values above.

Animation from https://en.wikipedia.org/wiki/Pascal%27s_triangle
The total number of paths is $2^n$.

If each turn to the right or left is equally probable, the probabilities of all paths are equal, and the probability of each path is:

$$p = \frac{1}{2^n} = 2^{-n}$$

The probability of a ball landing in bucket $k$ is the number of paths to the bucket multiplied by the probability of each path:

$$p(k) = \frac{n!}{k!(n-k)!} \cdot 2^{-n}$$
For a 7-row plinko, with 8 buckets labeled 0 to 7, what is the probability of a ball landing in bucket 1?

(There’s a hard way and an easy way!)

A) \( \sim 0.01 \)

B) \( \boxed{\sim 0.05} \)

C) \( \sim 0.1 \)

D) \( \sim 0.15 \)

E) \( \sim 0.2 \)

\[
p(1) = \frac{n!}{k!(n-k)!} \cdot 2^{-n} = \frac{7!}{1!(7-1)!} \cdot 2^{-7} = \frac{7!}{6!} \cdot 2^{-7} = 7 \cdot 2^{-7}
\]
What if the Plinko is Biased?

- Suppose that each peg in the plinko has been “fixed”, so that the probability of a left turn is 0.4 and the probability of a right turn is 0.6.

- For each of the paths to bucket $k$, there are $k$ right turns and $(n - k)$ left turns.

- For each individual path to bucket $k$, the probability is:

$$0.6^k \times 0.4^{(n-k)}$$

- The total probability of a ball falling in bucket $k$ is:

$$p(k) = \frac{n!}{k!(n-k)!} \times 0.6^k \times 0.4^{(n-k)}$$
Biased pegs “push” balls to the right.

Probability (number of paths) “draws” balls to the center.

Can you think of physical processes like this?
The Binomial Probability Distribution Function

- The general formulation:
  \( p(k; n, p) \) is the probability of \( k \) successes in \( n \) successive binary (yes/no) trials when the probability of success in each trial is \( p \).

- The probability function:

  \[
  p(k; n, p) = \frac{n!}{k!(n-k)!} p^k (1 - p)^{(n-k)}
  \]

- Some applications beyond plinkos:
  - Number of heads in \( n \) successive coin tosses.
  - Number of successes in prescribing a medication to a series of patients with the same condition.
  - Probability of surviving \( n \) potentially deadly events.
    \( p(n; n, p) \), were \( p \) is the probability of surviving each event.
Suppose that I let you put a ball in the 6-row plinko, and I agree to pay you $k$ dollars if the ball lands in bucket $k$.

This is probably going to cost me money!

How much should I charge you to play?

How much, on average, am I going to have to pay?
Clicker Question #3

How much should I charge you to play my plinko game (to break even)?
All answers count for now.

A) $1
B) $2
C) **$3**
D) $4
E) $6
F) $7
An Quick Solution

- Buckets 0 and 6 have equal probabilities. The average payout for these two is $3.
- Buckets 1 and 5 have equal probabilities. The average payout for these two is $3.
- Buckets 2 and 4 have equal probabilities. The average payout for these two is $3.
- The payout for bucket 3 is $3.
- The overall average payout is $3.

Without knowing any of the actual probabilities!
Random Variables

Definition: A variable that is assigned a value for each possible outcome or event for a probabilistic process.

Examples:

- For a coin toss, we could assign a random variable, $x$, the value of 1 for heads or 0 for tails.
- For $n$ successive coin tosses, we could define $x$ to be the number of heads.
- For the Plinko, we can define the random variable, $x$, as the number of the bucket that the ball lands in.
  But, we could define other random variables, too.